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Proactive Scheduling of Batch Processes by a Combined Robust Optimization and Multiparametric Programming Approach

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We address short-term batch process scheduling problems contaminated with uncertainty in the data. The mixed integer linear programming (MILP) scheduling model, based on the formulation of Ierapetritou and Floudas, Ind Eng Chem Res. 1998; 37(11):4341–4359, contains parameter dependencies at multiple locations, yielding a general multiparametric (mp) MILP problem. A proactive scheduling policy is obtained by solving the partially robust counterpart formulation. The counterpart model may remain a multiparametric problem, yet it is immunized against uncertainty in the entries of the constraint matrix and against all parameters whose values are not available at the time of decision making. We extend our previous work on the approximate solution of mp-MILP problems by embedding different uncertainty sets (box, ellipsoidal and budget parameter regulated uncertainty), and by incorporating information about the availability of uncertain data in the construction of the partially robust scheduling model. For any parameter realization, the corresponding schedule is then obtained through function evaluation. © 2013 American Institute of Chemical Engineers *AIChE J*, 59: 4184–4211, 2013
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Introduction

The area of scheduling of chemical and pharmaceutical processes has received significant attention in industry and academia with a number of excellent reviews summarizing the key contributions in this field.[1–8] Scheduling is likely to be subject to uncertainty attributed to endogenous factors such as varying processing times or production rates, as well as to exogenous factors arising from variations in the market demand, product prices, or time horizon, etc. A classification of sources of uncertainty in process operations is found in the work by Pistikopoulos.[9] The optimal scheduling policy for a chemical process based on nominal data may not be optimal or even feasible any more once a deviation from the nominal values has occurred. Proactive scheduling is motivated by the need to address uncertainty upfront in order to restrict disruptions and avoid rescheduling in response to disturbances. Provided knowledge about the probability distribution of the uncertain data is available, stochastic programming methods are widely used in proactive scheduling. In this category fall the works of Bonfill et al.,[10] Vin and Ierapetritou,[11] and Balasubramanian and Grossmann[12] addressing demand uncertainty, as well as Bonfill et al.,[13] Bonfill et al.,[14] and Balasubramanian and Grossmann[15] addressing operational level related uncertainty such as processing time variability. Scenario based formulations suffer from an increased problem size with respect to a growing number of uncertain parameters involved. In the open literature, another approach to account for the presence of uncertainty in the model is to employ its robust counterpart formulation. The objective of robust optimization is to identify scheduling policies that are feasible for all possible realizations of the parameters, or that meet an anticipated level of performance. Robust optimization is readily applicable to various types of uncertainty, affecting both the coefficients of the objective function as well as the constraints in the optimization problem. In the work of Lin et al.[16] and Janak et al.,[17] a robust counterpart formulation for mixed integer linear programming (MILP) models based on bounded uncertainty and known distribution, respectively, is presented. The robust model is then applied to short term batch process scheduling with price, demand, and processing time uncertainty. Comprehensive studies to derive the robust counterpart formulation of continuous and mixed integer multiparametric linear problems for different types of uncertainty sets have been conducted by Li and Ierapetritou,[18] and by Li et al.[19] The latter work also investigates combinations of well-studied uncertainty sets such as interval, polyhedral, and ellipsoidal sets and the applicability of the induced robust model to scheduling of batch processes.

Multiparametric programming is an analytical solution method.[20] It expresses the optimal or a close to optimal solution as a function of all uncertain parameters, which is found by exploring the parameter space through suitable strategies. It is a strong tool for optimization under uncertainty when the true values of the parameters are known to be available at the time of decision making.[21] Multiparametric programming has

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already found applications in process scheduling under uncertainty. Ryu et al.[22] and Ryu and Pistikopoulos[23] studied the special class of sequential processes with varying processing times and equipment availability. They derive a multiparametric (mp)-MILP scheduling problem that exhibits parameter dependent entries of the right-hand side (RHS) constraint vector. For the explicit solution of the mp-MILP model, the algorithm by Dua et al.[24] was employed. In the work of Li and Ierapetritou,[25] a framework to identify the parametric scheduling policy and the region where it is optimal around an initial parameter value was presented. The authors further explore the applicability of multiparametric programming in reactive scheduling as an alternative to reduce the response time adjusting the schedule to rush order or machine breakdown incidents.[26]

In this work, we first present a classification of the various types of uncertainty according to their availability at the time of decision making in the scheduling process. A combined robust optimization and multiparametric programming approach is then employed to generate a proactive scheduling strategy for short-term batch processes. A partially robust multiparametric counterpart problem of the scheduling model is propagated that also depends on the structure of the uncertainty set. We show that this approach allows for the treatment of various types of uncertainty at multiple locations in the underlying mathematical model, and remains flexible toward the incorporation of data once their actual values are known.

In the next section, the short-term batch process scheduling formulation, a mixed integer linear model, contaminated with uncertainty, and a two-stage method for the approximate solution of mp-MILP problems are introduced. This is followed by investigating several case studies highlighting the applicability of our approach in proactive scheduling.

Scheduling Under Uncertainty

Scheduling formulation

We consider network-represented short-term scheduling of batch processes. In particular, a unit specific event based model, which is a continuous time formulation featuring the concept of event points, is used.[27–29] Event points are time related instances, assigning starting and finishing times of tasks for each unit. The number of event points determines the maximum number of tasks that may be performed in each unit within the time horizon. Binary variables that are introduced into the model indicate the activation status of a task at a unit at an event point. The objective in scheduling is the maximization of profit although other criteria such as the minimization of make-span may also be considered. The deterministic scheduling formulation, based on the work of Ierapetritou and Floudas,[27] reads as follows:

$$\max_{x_{sn},st_{sn},sti_s,b_{ijn},t^s_{ijn},t^f_{ijn},w_{ijn}} \left(\sum_{s\in S,n\in N} P_s x_{sn} - \sum_{s\in S} C_s sti_s \right)$$

$$\sum_{i\in I_j} w_{ijn} \le 1 \qquad j\in J, n\in N \tag{1}$$

$$st_{sn} = sti_s - x_{sn} - \sum_{i\in I_s} \rho^c_{si} \sum_{j\in J_i} b_{ijn} \qquad s\in S, n=1 \tag{2}$$

$$st_{sn} = st_{s(n-1)} - x_{sn} - \sum_{i\in I_s} \rho^c_{si} \sum_{j\in J_i} b_{ijn} + \sum_{i\in I_s} \rho^p_{si} \sum_{j\in J_i} b_{ij(n-1)} \qquad s\in S, n>1, n\le N^{\max} \tag{3}$$

$$st_{sn} \le ST^{\max}_s \qquad s\in S, n\in N \tag{4}$$

$$V^{\min}_{ij} w_{ijn} \le b_{ijn} \le V^{\max}_{ij} w_{ijn} \qquad i\in I, j\in J_i, n\in N \tag{5}$$

$$\sum_{n\in N} x_{sn} \ge R_s \qquad s\in S \tag{6}$$

$$t^f_{ijn} \ge t^s_{ijn} + \alpha_{ij} w_{ijn} + \beta_{ij} b_{ijn} \qquad i\in I, j\in J_i, n\in N \tag{7}$$

$$t^s_{ij(n+1)} \ge t^f_{ijn} - H\left(1-w_{ijn}\right) \qquad i\in I, j\in J_i, n\ge 1, n<N^{\max} \tag{8}$$

$$t^s_{ij(n+1)} \ge t^f_{i'jn} - H\left(1-w_{i'jn}\right) \quad i,i'\in I_j, i\ne i', j\in J, n\ge 1, n<N^{\max} \tag{9}$$

$$t^s_{ij(n+1)} \ge t^f_{i'j'n} - H\left(1-w_{i'j'n}\right) \quad i\in I_j, i'\in I_j', i\ne i', j,j'\in J, j\ne j', n\ge 1, n<N^{\max} \tag{10}$$

$$t^s_{ij(n+1)} \ge t^s_{ijn}, t^f_{ij(n+1)} \ge t^f_{ijn} \qquad i\in I, j\in J_i, n\ge 1, n<N^{\max} \tag{11}$$

$$t^s_{ijn} \le H, t^f_{ijn} \le H \qquad i\in I, j\in J_i, n\in N \tag{12}$$

$$x_{sn}, st_{sn}, sti_s, b_{ijn}, t^s_{ijn}, t^f_{ijn} \ge 0 \qquad i\in I, j\in J_i, n\in N$$

$$w_{ijn}\in\{0,1\} \qquad i\in I, j\in J_i, n\in N$$

where S is the index set for all raw materials, intermediate and final products, I is the index set for all task, and J is the index set for all available units. N is the number of event points chosen. A full nomenclature is provided as a Table at the end.

In the scheduling formulation, allocation constraints Eq. 1 ensure that at any event point at most one task may be performed in a unit. The model accounts for material balances of all raw, intermediate and final products consumed and produced by the tasks involved, which is represented by constraints Eqs. 2 and 3. Storage restrictions for materials, as well as capacity limitations to perform a task in a particular unit are enforced by constraints Eqs. 4 and 5. Market demands, expressed by constraints Eq. 6, must also to be met. Constraints Eq. 7 are duration constraints. Sequencing constraints enforce that a new task in any unit may only start once all previous ones in that unit have finished. The constraints Eq. 8 address the issue of the same task taking place in the same unit. The constraints Eq. 9 account for different tasks taking place in the same unit. Furthermore, the sequencing constraints Eq. 10 are production recipe dependent. They account for different tasks taking place in different units that require to be performed consecutively. All tasks must have started and finished within the time horizon, modeled by constraints Eqs. 11 and 12.

The above scheduling formulation is an MILP model. The presence of uncertainty may transform the scheduling problem into the general multiparametric mp-MILP problem (P),

$$(P)\begin{cases} z(\theta) & := \min_{x,y}\left((c+H\theta)^T x + (d+L\theta)^T y \right) \\ \text{s.t.} & A(\theta)x + E(\theta)y \le b + F\theta \\ & x\in\mathbb{R}^n, y\in\{0,1\}^p \\ & \theta\in\Theta\subseteq\mathbb{R}^q \end{cases}$$

where θ denotes the vector of parameters and $c \in \mathbb{R}^n$, $H \in \mathbb{R}^{n\times q}$, $d \in \mathbb{R}^p$, $L \in \mathbb{R}^{p\times q}$, $A(\theta) \in \mathbb{R}^{m\times n}$, $E(\theta) \in \mathbb{R}^{m\times p}$, $b \in \mathbb{R}^m$, and $F \in \mathbb{R}^{m\times q}$. The matrices $A(\theta)$ and $E(\theta)$ are affine mappings with respect to θ. We have

$$A(\theta) := A^N + \sum_{l=1}^{q} \theta_l A^l$$

with $A^N \in \mathbb{R}^{m\times n}$, $A^l \in \mathbb{R}^{m\times n}$ for all l, where A^N denotes the nominal part of $A(\theta)$. Matrix $E(\theta)$ is defined analogously.

In (P), the vector x concatenates all continuous optimization variables from the scheduling formulation, whereas y represents the binary variables w_{ijn}. The parameters are introduced to model exogenous and endogenous data variability induced by price, demand, or operational level related uncertainty.

Classification of uncertainty

Given the scheduling formulation (P), we distinguish between parameters that are known to become available at some future time, that is, the exact value is revealed, and those that still remain unknown at the time of decision making. We define

$$\Theta := \Theta^1 \times \Theta^2,$$

where Θ^1 represents the set of revealing parameters, and Θ^2 represents the set of parameters with values unknown at the decision stage for which there is a need to immunize against in the model.

This classification of uncertain data is relevant for the solution of (P) and, consequently, in the construction of the proactive scheduling policy. Multiparametric programming as analytical solution method is a powerful tool to account for the presence of uncertainty in mathematical models, provided that the values of the parameters are known at the time of decision making.[21] On the other hand, Robust optimization identifies feasible solutions that are immune to variations in the data, making it independent of the outcome of the true values. The treatment of nonrevealing parameters from Θ^2 clearly benefits from a robust optimization approach, whereas multiparametric programming may be a preferred strategy for Θ^1.

Two-stage method for general mp-MILP problems

A variety of multiparametric programming algorithms is available for special classes of (P) with RHS and/or objective function coefficient (OFC) uncertainty,[24,30–36] whereas the presence of left-hand side (LHS) uncertainty in the model poses a particular challenge for its analytical solution. The optimal solution of (P) is a possibly discontinuous piecewise fractional polynomial function, and corresponding critical regions, subsets of the parameter space for which a particular solution remains optimal, are not necessarily convex sets.[37–39] These properties, in combination with the discrete nature of the problem, make it difficult to explore the parameter space efficiently to identify the optimal solution of (P). For the approximate solution of the general mp-MILP problem (P), a two-stage method has been proposed.[40] Problem (P) is immunized against LHS-uncertainty using a worst-case oriented approach for box constrained uncertainty, yielding a partially robust mp-MILP counterpart formulation. The partially robust model is then solved with a suitable multiparametric programming algorithm. The resulting optimal partially robust solution exhibits the favorable property of being a piecewise affine, albeit not necessarily continuous, function.

Here, we extend the framework of the two-stage method to solve (P) by encompassing uncertainty set induced partially robust models based on box constrained uncertainty, budget parameter regulated uncertainty and ellipsoidal uncertainty, respectively. Following the classification of the parameters as described above, two main steps are performed:

Step 1—Approximation Stage. By defining $\Theta := \Theta^1 \times \Theta^2$, the approximate solution of (P) is understood to be a function of $\theta^1 \in \Theta^1$ that, at the same time, is a feasible solution for every $\theta^2 \in \Theta^2$. Thus, it is the optimal solution of the following partially robust counterpart (RC') of (P),

$$(RC')\begin{cases} r(\theta^1) & := \min_{x,y} \max_{\theta^2\in\Theta^2} \big((c+H^1\theta^1+H^2\theta^2)^T x + \\ & \quad (d+L^1\theta^1+L^2\theta^2)^T y\big) \\ \text{s.t.} & \max_{\theta^1\in\Theta^1} \big([a_i^1(\theta^1)]^T x) + [e_i^1(\theta^1)]^T y\big) \\ & \quad + \max_{\theta^2\in\Theta^2} \big([a_i^2(\theta^2)]^T x + [e_i^2(\theta^2)]^T y\big) \\ & \quad \le b_i + [f_i^1]^T \theta^1 + \min_{\theta^2\in\Theta^2} [f_i^2]^T \theta^2, \\ & \quad i=1,\ldots,m \\ & x \in \mathbb{R}^n, \quad y \in \{0,1\}^p, \quad \theta^1 \in \Theta^1 \end{cases}$$

where subscript i denotes the i-th row of a matrix and $[\cdot]$ denotes a column vector. At this stage, (RC') is a multiparametric min-max problem.

Problem (RC') is then transformed into an mp-MILP problem (RC),

$$(RC)\begin{cases} r(\theta^1) & := \min_{x',y'} \big((c'+H'\theta^1)^T x' + (d'+L'\theta^1)^T y'\big) \\ \text{s.t.} & A'x' + E'(\theta^1)y' \le b' + F'\theta^1 \\ & x' \in \mathbb{R}^n, y' \in \{0,1\}^p \\ & \theta^1 \in \Theta^1, \end{cases}$$

where the apostrophe indicates that (RC) differs from the original problem (P), involving transformed constraints and possibly auxiliary variables. The partially robust counterpart problem (RC) is constructed to be an mp-MILP problem with OFC-uncertainty, RHS-uncertainty, and LHS-uncertainty affecting entries of the constraints that are affiliated with the binary variables only. In the construction of (RC), the uncertainty sets Θ^1 and Θ^2 play a key role. We allow the uncertainty sets to be either box constrained, budget parameter regulated box constrained, or ellipsoidal. The mathematical formulation of the uncertainty set dependent partially robust models (RC) is derived in Appendix A.

Step 2—Optimization Stage. In Appendix B, we revisit the steps of a decomposition algorithm suitable for the solution of mp-MILP problems of type (P) when the constraint matrix A is independent of θ. The partially robust scheduling formulation (RC) is then solved by this algorithmic procedure.

Note that for the solution of (RC) the initial feasible set of parameters is divided into polyhedral convex critical regions. Each region may contain several candidate solutions, each associated with different realizations of integer variables. These are all stored, defining the *envelope of parametric profiles*. For every parameter point, the approximate policy is identified through function evaluation of the corresponding objective function values, selecting the one with the best performance index.

The flowchart of the procedure is given in Figure 1. In the next section, we present the suitability of the proposed method as a proactive scheduling strategy.

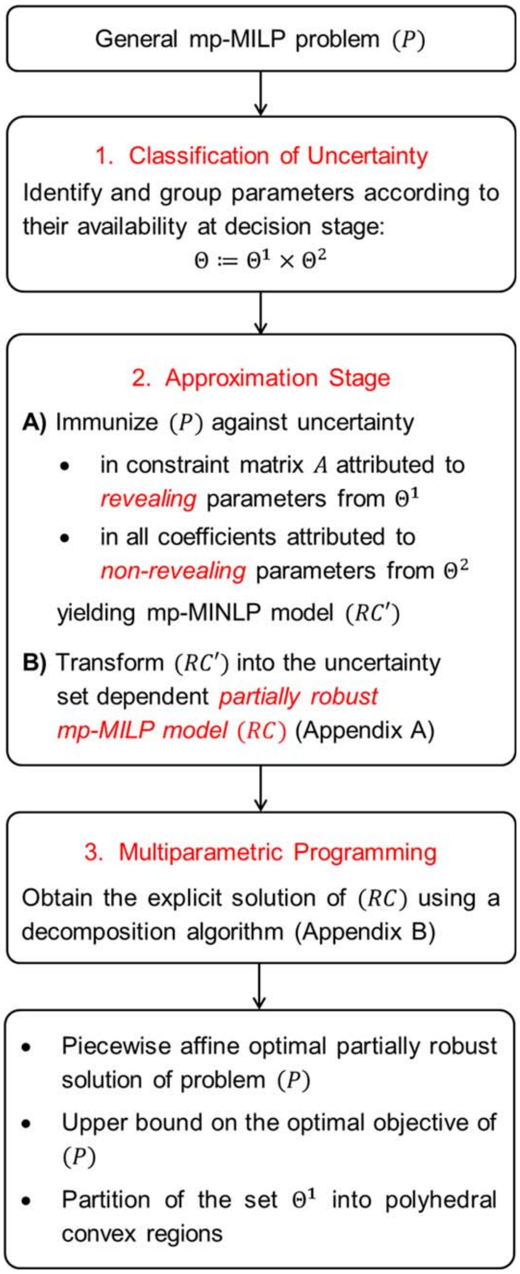

Figure 1. Steps of the proposed method for the approximate solution of the general mp-MILP problem (*P*).

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Proactive Short-Term Scheduling of Batch Processes

Case studies to illustrate the potential and characteristics of the combined robust optimization and multiparametric programming approach are taken from Ierapetritou and Floudas[27] and Wu and Ierapetritou.[41] They are altered to include data variability at the optimization stage. Example 1 involves a scheduling problem of small size, whereas Examples 2–4 are medium size scheduling problems and Example 5 presents a larger sized instance. Examples 1–3 feature price and demand variability. Additionally, Example 1 is affected by processing time uncertainty, Example 2 by conversion rate uncertainty, and Example 3 by time horizon uncertainty. Examples 4 and 5 are used to demonstrate that the approach is able to cope with a larger pool of parameters affecting prices, demands, storage capacities and operational level related data.

For each example, different uncertainty sets describing the parameter range are studied. We assume box constrained uncertainty sets for Examples 1, 3, and 5, box constrained and budget parameter regulated box constrained uncertainty sets for Example 2, and box constrained and ellipsoidal uncertainty sets for Example 4. In addition, we monitor the impact on the conservatism of the proactive scheduling policies that may arise if parameters from an uncertainty set are expected to become available or, on the other hand, are unknown the decision stage.

We emphasize the main aspects in the construction of the partially robust scheduling model for each case study performed. For the parameterized scheduling policies, the characterization of the critical regions and the achieved profit is given and, as appropriate, depicted. The proactively generated policies are then evaluated at a fixed feasible parameter point and their properties are discussed. As a reference, the optimal scheduling policy for the nominal problem is also provided.

Example 1. A production process, with a corresponding state task network (STN) representation shown in Figure 2, consists of three tasks that take place in three separate units. The final product S3 and its purified version, S4, are sold off to the market. The relevant data for Example 1 are presented in Tables 1 and 2. We assume price, demand, and processing time uncertainty for the mixing task inflicted by two parameters. The unit specific event based scheduling model features five event points over a time horizon of 12 h.

Case A. The parameters are box constrained, belonging to the following uncertainty set

$$\Theta^1_\infty :=\left\{\theta \in \mathbb{R}^2 | -0.75 \le \theta_1 \le 0.75, -0.5 \le \theta_2 \le 0.5\right\}.$$

For the nominal value, $\theta^N=(0,0)^T$, the Gantt-chart of the optimal scheduling policy is given in Figure 3. In the Gantt-chart, the intervals represent the duration a unit is engaged. Each interval is labeled with the task that is performed and also depicts the batch size that is processed by it in the corresponding unit. The overall profit of the nominal problem is 74.88, selling 48.1 and 55 units of S3 and S4, respectively, to the market.

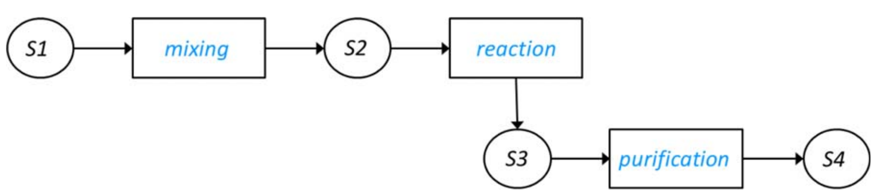

Figure 2. STN Representation of Example 1.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

Table 1. Data for Example 1.

Unit	Capacity	Task	Processing Time τ_{ij}
U1	100	Mixing	$4.5+\theta_2$
U2	75	Reaction	3
U3	50	Separation	1.5

Price variability introduces OFC-uncertainty, whereas demand variability introduces RHS-uncertainty into the scheduling formulation. Processing time variability introduces LHS-uncertainty in the model. Employing the two-stage method for the approximate solution of Example 1, the partially robust model enforces feasibility of the scheduling strategy for all scenarios of the processing time of the mixing tasks, in particular, with respect to the variable processing time per batch size only. The corresponding partially robust duration constraints induced by Θ^1_∞ read as follows

$$t^f_{\text{mix},\text{U1},n} \geq t^s_{\text{mix},\text{U1},n} + \tfrac{2}{3}(4.5+\theta_2)w_{\text{mix},\text{U1},n} + \tfrac{2}{300}5b_{\text{mix},\text{U1},n}, \quad n=1,\ldots,5.$$

The demand for the final product S4 is given by

$$\sum_{n=1}^{5} x_{\text{S4},n} \geq 55-20\theta_1.$$

The critical regions as obtained with the two-stage method are given in Figure 4. In all but one region a single solution is stored, which is the approximate scheduling policy. In region CR_2, two solutions are stored. The anticipated profit is given in Table 3.

We consider the parameter point $\theta^*=(0.75,-0.5)^T \in \Theta^1_\infty$, which belongs to the region CR_2. The Gantt-chart for the two profiles stored in CR_2 as obtained by the two-stage method are given in Figures 5 and 6, respectively. Although both are feasible policies by construction, the policy associated with the second profile is the optimal partially robust scheduling policy. It yields $z^2=128.6$, delivering 40.8 units of S3 and 66.7 units of S4 to the market, compared to an anticipated profit $z^1=116$ from selling 59.9 and 50 units of S3 and S4, respectively, for the first profile. The higher profit makes the second profile the propagated approximate schedule for θ^*. Note that the optimal scheduling policy for the nominal value of the parameters violates the duration constraints for the mixing task at this point.

Case B. Assume that the processing times will not be available at the time of decision making. They belong to the class of nonrevealing parameters. The uncertainty sets are given by

$$\Theta^1_\infty := \{\theta_1 \in \mathbb{R} | -0.75 \leq \theta_1 \leq 0.75\}$$

and

$$\Theta^2_\infty := \{\theta_2 \in \mathbb{R} | -0.5 \leq \theta_2 \leq 0.5\}.$$

The robustified duration constraints embedded in the partially robust model are

$$t^f_{\text{mix},\text{U}1,n} \geq t^s_{\text{mix},\text{U}1,n} + \frac{2}{3}5w_{\text{mix},\text{U}1,n} + \frac{2}{300}5b_{\text{mix},\text{U}1,n}, \quad n=1,\ldots,5.$$

The approximate scheduling policy is now independent of θ_2. The envelope of profits obtained with the two-stage method is given in Table 4. Note that the corresponding approximate scheduling policy is feasible for the case that θ_2 is known to be available at the time decision making. The critical regions and the envelope of profits are depicted in Figure 7. The partially robust model with respect to Case B is more conservative than for Case A for every parameter realization. For example, at the point $\theta^*=(0.75,-0.5)^T$ the optimal partially robust profit is $z^2=103.8$, selling 23.2 units of S3 and 58.4 units of S4, associated with the second profile stored in region CR_2. The Gantt-chart of the approximate scheduling policy at this point is given in Figure 8.

Case C. Assume that none of the parameters becomes available at the time of decision making, that is

$$\Theta^2_\infty := \{\theta \in \mathbb{R}^2 | -0.75 \leq \theta_1 \leq 0.75, -0.5 \leq \theta_2 \leq 0.5\}.$$

The partially robust model induced by Θ^2_∞ reduces to a deterministic problem comprising of the worst case scenarios with respect to the duration constraints of the mixing task and the demand of product S4,

$$t^f_{\text{mix},\text{U1},n} \geq t^s_{\text{mix},\text{U1},n} + \tfrac{2}{3}5w_{\text{mix},\text{U1},n} + \tfrac{2}{300}5b_{\text{mix},\text{U1},n}, \quad n=1,\ldots,5$$

and

$$\sum_{n=1}^{5} x_{\text{S4},n} \geq 70,$$

respectively. In the worst case, the price for S4, P_{S4}, is zero. The robust counterpart formulation of Example 1 given the uncertainty set Θ^2_∞ is infeasible.

Example 2. The process involves the production of two final products and several intermediate products as depicted in the STN-representation, Figure 9. The reaction tasks denoted as R1, R2, and R3, respectively, take place in one of two units, U1 and U2. Units U3 and U4 are suitable for the heating and the separation task, respectively. The data for Example 2 is given in Tables 5 and 6. The prices and the demands of the final products S8 and S9 vary. There is uncertainty in the production rates of S8 and the intermediate product S7 as follows,

$$\rho^{pN}_{\text{S7,R3}} = 0.9, \quad \rho^{pR}_{\text{S7,R3}} = 0.1$$

and

$$\rho^{pN}_{\text{S8,R2}} = 0.35, \quad \rho^{pR}_{\text{S8,R2}} = 0.05$$

where the superscripts N and R denote the nominal value and the range of the rates, respectively. Uncertain production

Table 2. Data for Example 1.

State	Storage Capacity	Initial Amount	Initial Cost	Price	Demand
S1	–	–	0	0	0
S2	100	0	0	0	0
S3	100	0	0	0.7	0
S4	–	0	0	$0.75+\theta_1$	$55-20\theta_1$

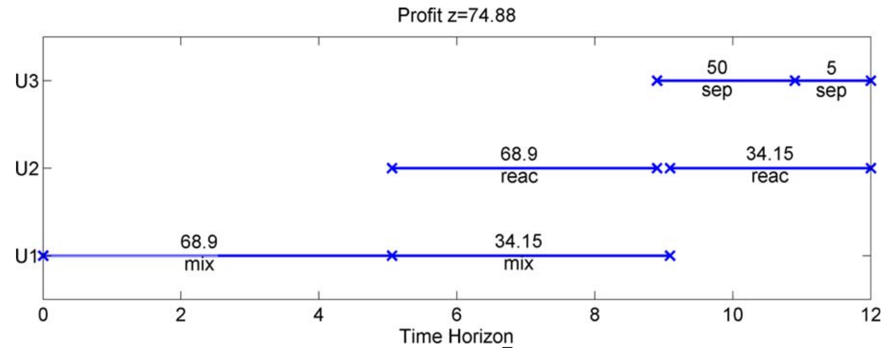

Figure 3. Gantt-chart for Example 1 at nominal value $\theta^N = (0, 0)^T$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

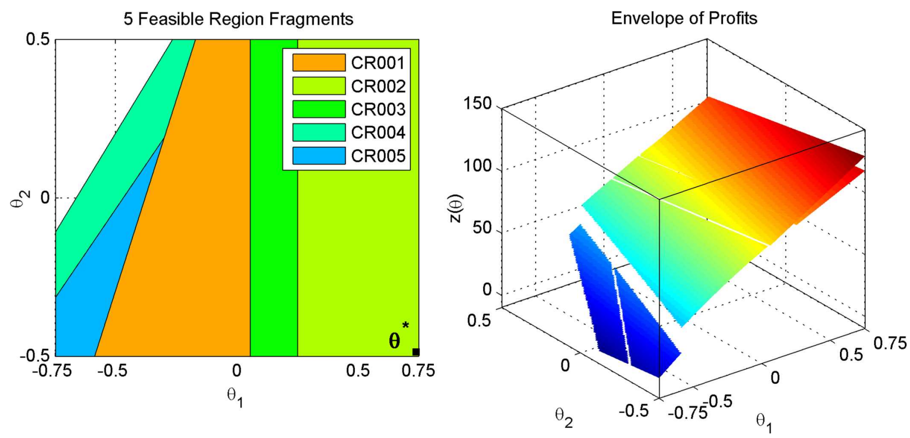

Figure 4. Critical regions and envelope of profits with two-stage method—Example 1, Case A.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

Table 3. Envelope of Profits with Two-Stage Method—Example 1, Case A.

	Critical Region	Envelope of Profits
CR_1	$\{-\theta_1+0.4\theta_2 \le 0.38, \theta_1 \le 0.05, -0.5 \le \theta_2 \le 0.5\}$	$-20\theta_1^2+56\theta_1-19\theta_2+69.7$
CR_2	$\{0.25 \le \theta_1 \le 0.75, -0.5 \le \theta_2 \le 0.5\}$	$z^1=50\theta_1-16\theta_2+71.4$ $z^2=-8.3\theta_1\theta_2+62.5\theta_1-18.5\theta_2+69.3$
CR_3	$\{0.05 \le \theta_1 \le 0.25, -0.5 \le \theta_2 \le 0.5\}$	$-8.3\theta_1\theta_2+62.5\theta_1-18.5\theta_2+69.3$
CR_4	$\{-\theta_1+0.79\theta_2 \le 0.66, \theta_1-0.9\theta_2 \le -0.46, -0.75 \le \theta_1, \theta_2 \le 0.5, \theta_1-0.4\theta_2 \le -0.37\}$	$-20\theta_1^2+60.2\theta_1-16\theta_2+54.6$
CR_5	$\{-\theta_1 \le 0.89\theta_2 \le 0.46, -0.75 \le \theta_1, \theta_1-0.4\theta_2 \le -0.3, -0.5 \le \theta_2\}$	$-20\theta_1^2+60.2\theta_1-16\theta_2+54.6$

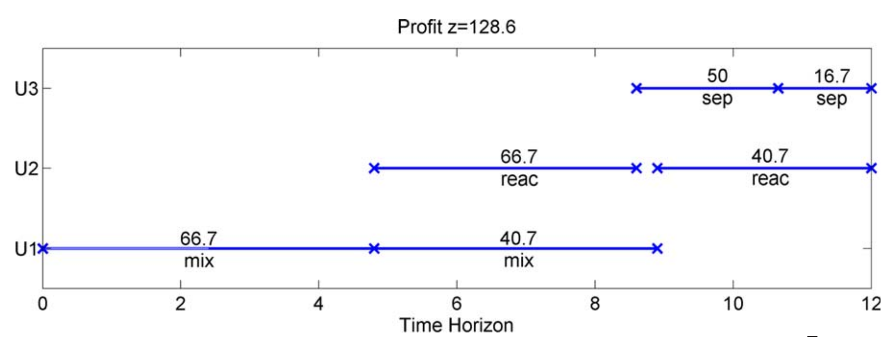

Figure 5. Gantt-chart for Example 1, Case A, with two-stage method at $\theta^* = (0.75, -0.5)^T \in CR_2$ with respect to the second profile stored in the envelope, which is the approximate scheduling policy at this realization.

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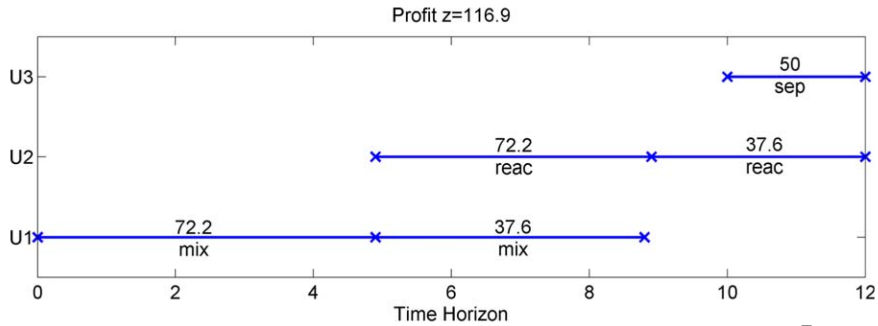

Figure 6. Gantt-chart for Example 1, Case A, with two-stage method at $\theta^* = (0.75, -0.5)^T \in CR_2$ with respect to the first profile stored in the envelope.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

Table 4. Envelope of Profits with Two-Stage Method—Example 1, Case B.

	Critical Region	Envelope of Profits
CR_1	$\{-0.17 \le \theta_1 \le 0.05, -0.5 \le \theta_2 \le 0.5\}$	$-20\theta_1^2+56\theta_1+60.2$
CR_2	$\{0.25 \le \theta_1 \le 0.75, -0.5 \le \theta_2 \le 0.5\}$	$z^1=50\theta_1+63.4$ $z^2=58.4\theta_1+60$
CR_3	$\{0.05 \le \theta_1 \le 0.25, -0.5 \le \theta_2 \le 0.5\}$	$58.4\theta_1+60$
CR_4	$\{-0.26 \le \theta_1 \le -0.17, -0.5 \le \theta_2 \le 0.5\}$	$-20\theta_1^2+60.2\theta_1+46.6$

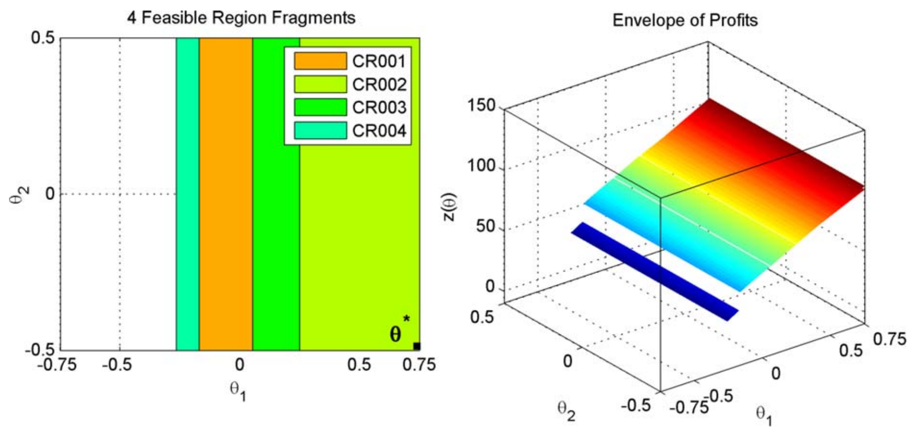

Figure 7. Critical regions and envelope of profits with two-stage method - Example 1, Case B.

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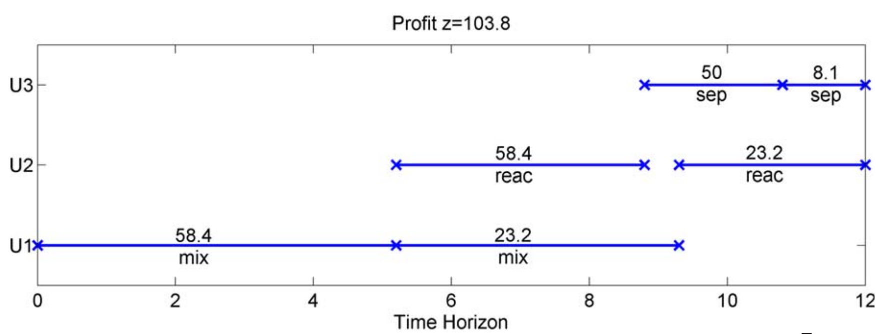

Figure 8. Gantt-chart for Example 1, Case B, with two-stage method at $\theta^* = (0.75, -0.5)^T \in CR_2$ with respect to the second profile stored in the envelope, which is the approximate scheduling policy at this realization.

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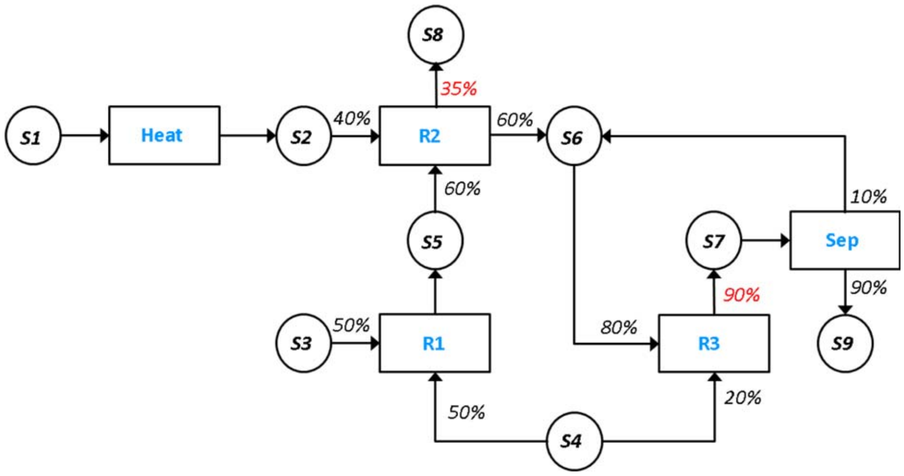

Figure 9. STN Representation of Examples 2, 3, and 4 showing nominal conversion rates.

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rates apply to all tasks that are involved in putting out S7 and S8, respectively, in any suitable unit.

While price and demand uncertainty are box constrained, the conversion rate uncertainty is modeled to belong to the uncertainty set with an adjustable degree of conservatism of the solution tuned by a budget parameter. Moreover, as conversion rate uncertainty affect only the entries of the constraint matrix affiliated with the continuous variables, without loss of generality, they can be classified as nonrevealing parameters. We introduce the parameters $\theta_{S7,l}, l=1,2$, associated with production of S7, to model uncertain conversion rate for reaction R3 in units U1 and U2, respectively. Analogously, $\theta_{S8,l}, l=1,2$, associated with production of S8, to model uncertain conversion rates for reaction R2 in units U1 and U2, respectively, are introduced.

The uncertainty sets are thus given by Θ^1_∞,

$$\Theta^1_\infty := \{\theta \in \mathbb{R}^2 | 0 \le \theta_l \le 1, l=1,2\},$$

which accounts for price and demand variability in the model, as well the sets $\Theta^2_{\Gamma_{S7}}$,

$$\Theta^2_{\Gamma_{S7}} := \{\theta_{S7} \in \mathbb{R}^2 | \exists S_{\Gamma_{S7}} \subseteq \{1,2\} : 0.8 \le \theta_{S7,l} \le 1 \ \forall l \in S_{\Gamma_{S7}}, \theta_{S7,l} = 0.9 \ \forall l \in \{1,2\} \backslash S_{\Gamma_{S7}}\},$$

and $\Theta^2_{\Gamma_{S8}}$

$$\Theta^2_{\Gamma_{S8}} = \{\theta_{S8} \in \mathbb{R}^2 | \exists S_{\Gamma_{S8}} \subseteq \{1,2\} : 0.3 \le \theta_{S8,l} \le 0.4 \ \forall l \in S_{\Gamma_{S8}}, \theta_{S8,l} = 0.35 \ \forall l \in \{1,2\} \backslash S_{\Gamma_{S8}}\}$$

with $\Gamma_{S7}, \Gamma_{S8} \in \{0,1,2\}$, reflecting the uncertain production rates. For task S7 and task S8, respectively, a budget parameter Γ_s is introduced. The choice of the budget parameters regulates how deviation of the production rates from the nominal value is supported. For $\Gamma_s=0$, the nominal production rates are attained. For $\Gamma_s=1$, deviation from the nominal value in at most one of the two units is assumed, whereas for $\Gamma_s=2$, the production rates in both units are expected to vary. The set $S_{\Gamma_s} \subseteq \{1,2\}$ denotes the index set with cardinality Γ_s, that is, $|S_{\Gamma_s}|=\Gamma_s$.

Table 5. Data for Examples 2, 3, and 4.

Unit	Capacity	Task	Processing Time τ_{ij}
U1	50	R1, R2, R3	2, 2, 1 (3+θ_5, 2, 1)
U2	80	R1, R2, R3	2, 2, 1
U3	100	Heating	1
U4	200	Separation	2

Conversion rate uncertainty affects the constraints accounting for material balances

$$\begin{aligned} st_{sn} &= sti_s - x_{sn} - \sum_{i \in I_s} \rho^c_{si} \sum_{j \in J_i} b_{ijn}, \quad s \in S, n=1 \\ st_{sn} &= st_{s(n-1)} - x_{sn} - \sum_{i \in I_s} \rho^c_{si} \sum_{j \in J_i} b_{ijn} \\ &\quad + \sum_{i \in I_s} \rho^p_{si} \sum_{j \in J_i} b_{ij(n-1)}, \quad s \in S, 2 \le n \le N^{\max}. \end{aligned} \tag{13}$$

Eq. 13 is a recursive formula and it may be written as

$$\begin{aligned} st_{sn} &= sti_s - x_{sn} - \sum_{i \in I_s} \rho^c_{si} \sum_{j \in J_i} b_{ijn}, \quad s \in S, n=1 \\ st_{sn} &= sti_s - x_{s1} - \sum_{i \in I_s} \rho^c_{si} \sum_{j \in J_i} b_{ij1} \\ &\quad + \sum_{2 \le n' \le n} \left(-x_{sn'} - \sum_{i \in I_s} \rho^c_{si} \sum_{j \in J_i} b_{ijn'} + \sum_{i \in I_s} \rho^p_{si} \sum_{j \in J_i} b_{ij(n'-1)} \right), \\ &\quad s \in S, 2 \le n \le N^{\max}. \end{aligned}$$

In the following, we will substitute the optimization variable st_{sn} in the scheduling formulation. It comprises of the amount of a state being produced, deducted by the amounts being consumed by any task or sold off to the market up to the point in time associated with event point n. The amount of any state must not exceed the maximum storage capacity and, naturally, be non-negative.

To derive the partially robust scheduling formulation, for every triplet snn' with $s \in S$, $2 \le n \le N^{\max}$, $2 \le n' \le n$, two budget parameters $\Gamma^A_{sn(n'-1)}$ and $\Gamma^B_{sn(n'-1)}$ are introduced. The partially robust storage constraints induced by the sets $\Theta^2_{\Gamma_{S7}}$ and $\Theta^2_{\Gamma_{S8}}$ read as follows

Table 6. Data for Examples 2, 3, and 4.

State	Storage Capacity	Initial Amount	Initial Cost	Price	Demand
S1	–	–	5 $(10-5\theta_1)$	0	0
S2	100	0	0	0	0
S3	–	–	5	0	0
S4	–	–	5	0	0
S5	150	0	0	0	0
S6	200 $(120+50\theta_3)$	0	0	0	0
S7	200 $(200+50\theta_4)$	0	0	0	0
S8	–	0	0	$10+10\theta_1$ $(10+5\theta_1)$	$50-40\theta_1$
S9	–	0	0	$15+5\theta_2$	$60-20\theta_2$ $(50+\theta_2)$

$$
\begin{aligned}
&sti_s - x_{sn} - \sum_{i\in I_s}\rho^c_{si}\sum_{j\in J_i} b_{ijn} \le ST^{\max}_s, \quad s\in\{S7,S8\}, n=1\\
&sti_s - x_{s1} - \sum_{i\in I_s}\rho^c_{si}\sum_{j\in J_i} b_{ij1} + \sum_{2\le n'\le n}\Bigg(-x_{sn'} - \sum_{i\in I_s}\rho^c_{si}\sum_{j\in J_i} b_{ijn'}\\
&\quad + \sum_{i\in I_s}\rho^{pN}_{si}\sum_{j\in J_i} b_{ij(n'-1)} + \Gamma^A_{sn(n'-1)} z^A_{sn(n'-1)} + \sum_{i\in I_s}\sum_{j\in J_i} p^A_{sn(n'-1)ij}\Bigg)\\
&\quad \le ST^{\max}_s, \quad s\in\{S7,S8\}, 2\le n\le N^{\max}\\
&z^A_{sn(n'-1)} + p^A_{sn(n'-1)ij} \ge \rho^{pR}_{si} b_{ij(n'-1)}, \quad s\in\{S7,S8\}, 2\le n\le N^{\max},\\
&\quad 2\le n'\le n, i\in I_s, j\in J_i\\
&z^A_{sn(n'-1)} \ge 0, p^A_{sn(n'-1)ij} \ge 0, \quad s\in\{S7,S8\}, 2\le n\le N^{\max},\\
&\quad 2\le n'\le n, i\in I_s, j\in J_i
\end{aligned}
\tag{14}
$$

with $z^A_{sn(n'-1)}$ and $p^A_{sn(n'-1)ij}$ being auxiliary optimization variables. The steps to derive the partially robust model for budget parameter regulated uncertainty are given in detail in Appendix A.

In a similar way, the robustified non-negativity constraints are formulated as

$$
\begin{aligned}
&sti_s - x_{sn} - \sum_{i\in I_s}\rho^c_{si}\sum_{j\in J_i} b_{ijn} \ge 0, \quad s\in\{S7,S8\}, n=1\\
&sti_s - x_{s1} - \sum_{i\in I_s}\rho^c_{si}\sum_{j\in J_i} b_{ij1} + \sum_{2\le n'\le n}\Bigg(-x_{sn'} - \sum_{i\in I_s}\rho^c_{si}\sum_{j\in J_i} b_{ijn'}\\
&\quad + \sum_{i\in I_s}\rho^{pN}_{si}\sum_{j\in J_i} b_{ij(n'-1)} - \Gamma^B_{sn(n'-1)} z^B_{sn(n'-1)}\\
&\quad - \sum_{i\in I_s}\sum_{j\in J_i} p^B_{sn(n'-1)ij}\Bigg) \ge 0, \quad s\in\{S7,S8\}, 2\le n\le N^{\max}\\
&z^B_{sn(n'-1)} + p^B_{sn(n'-1)ij} \ge \rho^{pR}_{si} b_{ij(n'-1)},\\
&\quad s\in\{S7,S8\}, 2\le n\le N^{\max}, 2\le n'\le n, i\in I_s, j\in J_i\\
&z^B_{sn(n'-1)} \ge 0, p^B_{sn(n'-1)ij} \ge 0,\\
&\quad s\in\{S7,S8\}, 2\le n\le N^{\max}, \quad 2\le n'\le n, i\in I_s, j\in J_i.
\end{aligned}
\tag{15}
$$

Note that uncertain coefficients of the constraint matrix introduced by conversion rate uncertainty are not independent. Nevertheless, a meaningful partially robust scheduling model is derived if multiple budget parameters per row are introduced as above, accounting for conversion rate uncertainty associated with different event points. To ensure consistency between different constraints of the same type for all event points, a sensible choice of the budget parameter is necessary. This is satisfied by setting

$$\Gamma^A_{sn(n'-1)} = \Gamma^A_{s\bar{n}(n'-1)}, \quad n\ne\bar{n}$$

and

$$\Gamma^B_{sn(n'-1)} = \Gamma^B_{s\bar{n}(n'-1)}, \quad n\ne\bar{n}.$$

Furthermore, the budget parameters introduced into the storage constraints and the budget parameters introduced into the non-negativity constraint need to have the same value, i.e.

$$\Gamma^A_{sn(n'-1)} = \Gamma^B_{sn(n'-1)}, \quad s\in\{S7,S8\}, 2\le n\le N^{\max}, 2\le n'\le n.$$

Remark 1. *If several tasks are involved in processing a state, then the partially robust storage and non-negativity constraints Eqs.* 14 *and* 15*, respectively, support the*

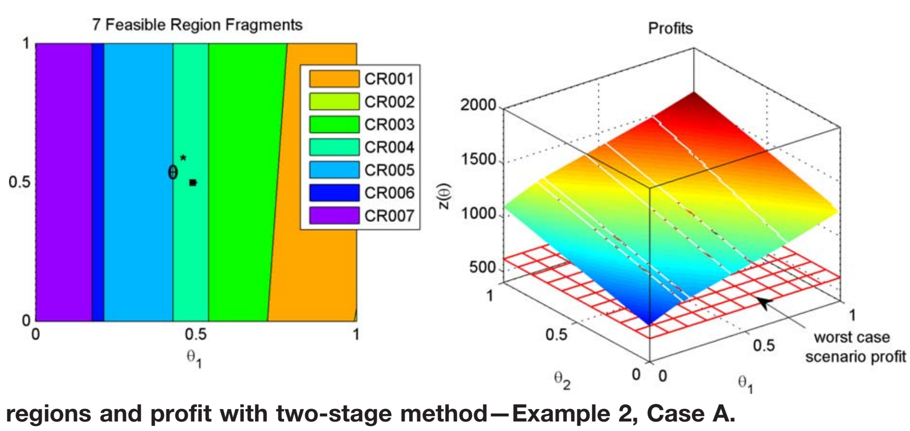

Figure 10. Critical regions and profit with two-stage method—Example 2, Case A.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

deviation of a certain number of productions rates, encompassing the rates for all task in all suitable units, from the nominal value. For a more rigorous scheme, i.e. uncertain production rates for individual tasks are budget parameter regulated, the budget parameters $\Gamma^{A,B}_{sn(n'-1)i}$ for every $i \in I_s$ are introduced instead. Auxiliary variables and the constraints require to be altered accordingly. In our example, products S7 and S8 are put out by a single task each, a distinction as above is therefore not relevant.

Budget parameter regulated uncertainty for consumption rates is modelled analogously.

The continuous time scheduling formulation of Example 2 features a choice of five event points over a time horizon of 8 h.

Case A. We assume that at most one of the production rates in U1 and U2 is likely to change from the nominal value. This is enforced by setting

$$\Gamma^{A}_{sn(n'-1)}=\Gamma^{B}_{sn(n'-1)}=1, \quad s \in \{S7, S8\}, 2 \le n \le 5, 2 \le n' \le n$$

in the corresponding constraints Eqs. 14 and 15. The partially robust scheduling formulation depends on θ_1 and θ_2 only. Seven critical regions are identified. The partition of the $\theta_1-\theta_2$ space along with the profit is depicted in Figure 10 and given in Appendix C, Table C1.

Case B. Setting

$$\Gamma^{A}_{sn(n'-1)}=\Gamma^{B}_{sn(n'-1)}=2, \quad s \in \{S7, S8\}, 2 \le n \le 5, 2 \le n' \le n$$

resembles the worst-case with respect to production rates in the partially robust model. As expected, its optimal solution is more conservative than that for Case A and an overall lower profit is achieved, see Figure 11. The critical regions with respect to Θ^1_∞ and the partially robust profit are presented in Appendix C, Table C2.

Cases C, D, and E. If no deviation from the nominal values of the production rates is supported, Case C, this is modelled by setting $\Gamma^{A,B}_{S7,S8}=0$. In Case D, we assume that at most one of the production rates processing S7 but both of the rates for S8 are likely to change from the nominal value. The corresponding partially robust model requires a choice of $\Gamma^{A,B}_{S7}=1$ and $\Gamma^{A,B}_{S8}=2$ for the budget parameters. In Case E, the reverse scenario is considered. Thus, $\Gamma^{A,B}_{S7}=2$ and $\Gamma^{A,B}_{S8}=1$ is used. The critical regions of the partially robust scheduling models along with the corresponding profits as obtained with the two-stage method for the Cases C, D, and E of Example 2 are given in Figures C1–C3, respectively, in Appendix C.

Case F. Finally, we consider the worst case scenario of Example 2 with respect to the given uncertainty sets $\Theta^1_\infty, \Theta^2_{\Gamma_{S7}}$, and $\Theta^2_{\Gamma_{S8}}$. All production rates are then likely to change from the nominal value, which is accounted for by setting $\Gamma^{A,B}_{S7,S8}=2$ in the robustified constraints Eqs. 14 and 15. The demand constraints for products S8 and S9 that are most difficult to maintain are given by

$$\sum_{n=1}^{5} x_{S8,n} \ge 50, \ \sum_{n=1}^{5} x_{S9,n} \ge 60,$$

which is attained at the point $\theta=(0,0)^T \in \Theta^1_\infty$. The prices for S8 and S9 are excepted to be at their lowest attainable values, $P_{S8}=10$ and $P_{S9}=15$, respectively. The same result is achieved if demand and price related parameters are treated as nonrevealing. For the worst-case scenario, the robust counterpart is a deterministic problem, yielding a profit of $z=615.3$. Note that the profit is a valid constant lower bound on the profits of the approximate solutions obtained in Cases A-E, see Figures 10 and 11 and Figures C1–C3 in Appendix C.

In Appendix C, we compare the schedules for the point $\theta^*=(0.5,0.5)^T \in \Theta^1_\infty$, $\theta^*_{S7}=(0.9,0.9)^T \in \Theta^2_{\Gamma_{S7}}$, and $\theta^*_{S8}=(0.35,0.35)^T \in \Theta^2_{\Gamma_{S8}}$, which corresponds to the nominal value of the parameters involved. Figures C4–C9 provide the propagated approximate scheduling policies for Example 2, Cases A–F, as obtained with the two-stage method. With exception of Case F, the schedules are obtained from function evaluation of the parametric profiles with respect to θ^*. It holds $\theta^* \in CR_4$ for Case A, $\theta^* \in CR_2$ for Case B, $\theta^* \in CR_1$ for Case C, $\theta^* \in CR_3$ for Case D, and $\theta^* \in CR_2$ for Case E. Note that the schedule for Case C at θ^* represents the optimal policy for the nominal scheduling problem.

Example 3. The same production process as in Example 2 with price and demand uncertainty is considered. Production rates for S7 and S8 do not deviate from their nominal values. However, the time horizon H for the production span is uncertain, but is expected to be known at the time of decision making. It is described by

$$H=8+2\theta_3, \quad 0 \le \theta_3 \le 1.$$

We use five event points for the continuous time scheduling formulation.

Case A. The uncertainty set for Example 3 is given by

$$\Theta^1_\infty := \{\theta \in \mathbb{R}^3 | 0 \le \theta_l \le 1, l=1,2,3\}.$$

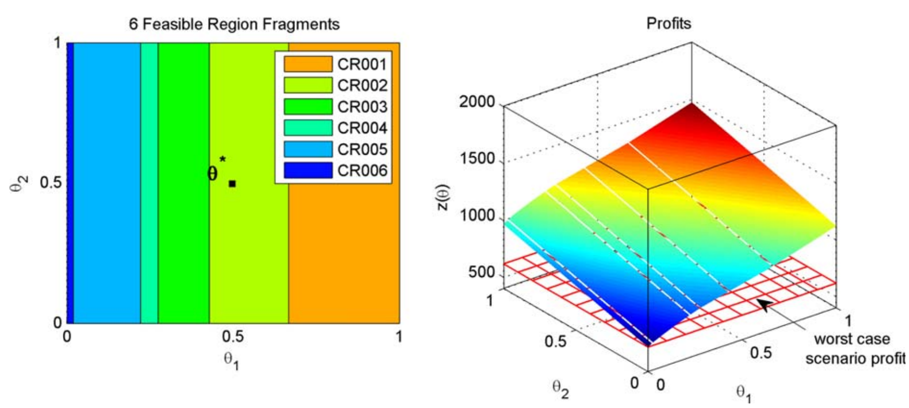

Figure 11. Critical regions and profit with two-stage method—Example 2, Case B.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

Note that price, demand and time horizon variability as above introduce OFC- and RHS-uncertainty, and LHS-uncertainty affecting the entries of the constraint matrix associated with binary variables only into the scheduling model. We define the index sets I and J,

$$I := \{\text{R1}, \text{R2}, \text{R3}, \text{Heat}, \text{Sep}\}, \quad J := \{\text{U1}, \text{U2}, \text{U3}, \text{U4}\},$$

for each task the index set J_i of units suitable for processing task i,

$$J_{\text{R1,R2,R3}} := \{\text{U1}, \text{U1}\}, \quad J_{\text{Heat}} := \text{U3}, \quad J_{\text{Sep}} := \text{U4},$$

and for each unit the index set I_j of all tasks performable in unit j,

$$I_{\text{U1,U2}} := \{\text{R1}, \text{R2}, \text{R3}\}, \quad I_{\text{U3}} := \text{Heat}, \quad I_{\text{U4}} := \text{Sep}.$$

Time horizon uncertainty occurs in sequencing constraints for the same task taking place in the same unit,

$$t^s_{ij(n+1)} \geq t^f_{ijn} - (8+2\theta_3)(1-w_{ijn}), \quad i \in I, j \in J_i, n=1,\ldots,4,$$

different tasks taking place in the same unit,

$$t^s_{ij(n+1)} \geq t^f_{i'jn} - (8+2\theta_3)(1-w_{i'jn}),$$
$$i, i' \in I_j, i \neq i', j \in J, n=1,\ldots,4,$$

and in constraints according to the process specific production recipe for successive tasks,

$$t^s_{ij(n+1)} \geq t^f_{i'j'n} - (8+2\theta_3)(1-w_{i'j'n})$$
$$i=R2, i'=\text{Heat}, j \in J_i, j' \in J_{i'}, n=1,\ldots,4$$
$$t^s_{ij(n+1)} \geq t^f_{i'j'n} - (8+2\theta_3)(1-w_{i'j'n})$$
$$i=R2, i'=R1, j \in J_i, j' \in J_{i'}, j \neq j', n=1,\ldots,4$$
$$t^s_{ij(n+1)} \geq t^f_{i'j'n} - (8+2\theta_3)(1-w_{i'j'n})$$
$$i=R3, i'=R2, j \in J_i, j' \in J_{i'}, j \neq j', n=1,\ldots,4$$
$$t^s_{ij(n+1)} \geq t^f_{i'j'n} - (8+2\theta_3)(1-w_{i'j'n})$$
$$i=\text{Sep}, i'=R3, j \in J_i, j' \in J_{i'}, n=1,\ldots,4.$$

RHS-uncertainty, present in the demand constraints for products S8 and S9,

$$\sum_{n=1}^{5} x_{\text{S8},n} \geq 50-40\theta_1, \quad \sum_{n=1}^{5} x_{\text{S9},n} \geq 60-20\theta_2,$$

additionally appears in the time limitations,

$$t^s_{ijn} \leq 8+2\theta_3, t^f_{ijn} \leq 8+2\theta_3, \quad i \in I, j \in J_i, n=1,\ldots,5.$$

The perturbed scheduling model is in agreement with its partially robust counterpart formulation. In this case, the two-stage method finds the optimal scheduling strategy. A total of 11 critical regions is identified for which a selection along with the corresponding optimal profit is given in Appendix C, Table C3. The Gantt-chart depicting the scheduling policy for the parameter realization $\theta^*=(0.5, 0.5, 0.5)^T \in CR_2$, which is the nominal value with respect to Θ^1_∞, is given in Figure 12.

Case B. We consider the case that $\theta_2=0$, fixing the price and demand of final product S9. The uncertainty set reduces to

$$\Theta^1_\infty := \{\theta \in \mathbb{R}^3 | 0 \leq \theta_l \leq 1, l \in \{1,3\}, \theta_2=0\}$$

Thus, the optimal scheduling policy depends on θ_1 and θ_3 only. The partition of the parameter space into 13 critical regions and the optimal profit as obtained with the two-stage method is depicted in Figure 14, and a selection of the regions is given in Appendix C, Table C4. The Gantt-chart of the identified policy for the parameter realization $\theta^*=(0.5, 0, 0.5)^T \in CR_4 \subseteq \Theta^1_\infty$ is given in Figure 13.

Example 4. We consider the same process as in Example 2. The data for Example 4 is given in Tables 5 and 6 where, if it differs from the data of Example 2, it is given in brackets. There are several parameters affecting prices, demands, storage capacities, and processing times of various states and tasks that are known to be available at the time of decision making. They are contained in the set Θ^1_∞,

$$\Theta^1_\infty := \{\theta \in \mathbb{R}^5 | -1 \leq \theta_l \leq 1, l=1,\ldots,5\}.$$

In addition, conversion rate uncertainty for the production of intermediate product S7 in the units U1 and U2 occurs. It is measurable, but known to belong to an ellipsoidal uncertainty set. The nominal value and the range of the production rate uncertainty is given by

$$\rho^{pN}_{\text{S7,R3}} = 0.9, \quad \rho^{pR}_{\text{S7,R3}} = 0.1.$$

Let $\theta_{\text{S7},1}$ and $\theta_{\text{S7},2}$ denote the parameters accounting for production rate uncertainty for S7 in unit U1 and unit U2, respectively. Production rate uncertainty introduces LHS-

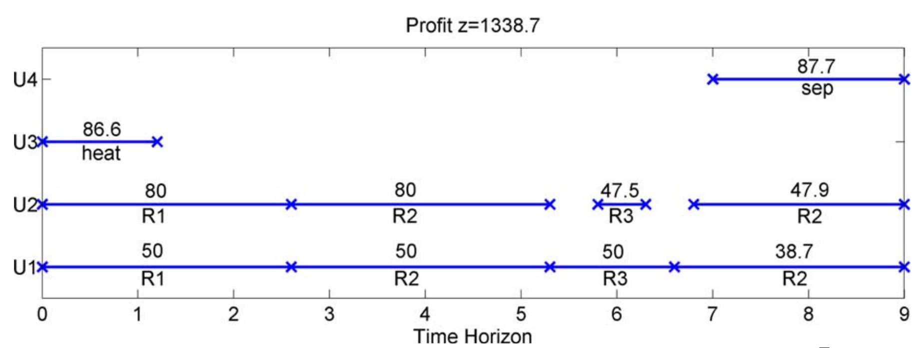

Figure 12. Gantt-chart for Example 3, Case A, with two-stage method at $\theta^*=(0.5, 0.5, 0.5)^T \in CR_2$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

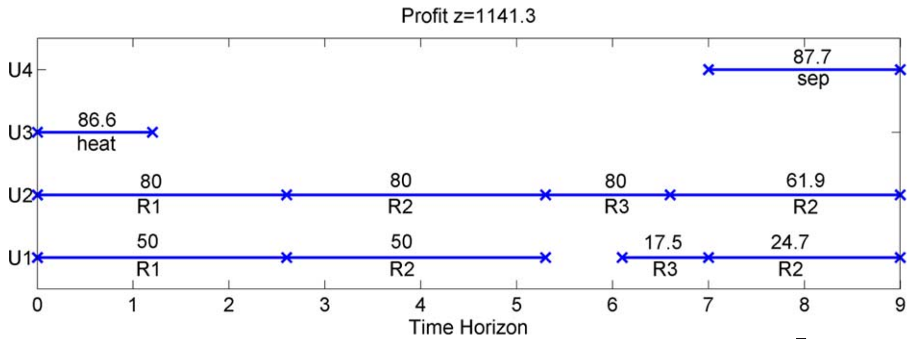

Figure 13. Gantt-chart for Example 3, Case B, with two-stage method at $\theta^*=(0.5, 0, 0.5)^T \in CR_4$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

uncertainty, affecting the entries of the constraint matrix A, into the model. Hence, it may likewise be regarded as nonrevealing uncertainty. The corresponding uncertainty set is then given by

$$\Theta_2^2 := \left\{ \theta_{\mathrm{S7}} \in \mathbb{R}^2 | \gamma_l := \frac{\theta_{S7,l} - \rho^{pN}_{\mathrm{S7,R3}}}{\rho^{pR}_{\mathrm{S7,R3}}}, l=1,2, ||\gamma||_2 \leq 1 \right\}. \tag{16}$$

The continuous time scheduling formulation features five event points over a time horizon of 8 h.

Case A. To apply the proposed method to Example 4, the ellipsoidal uncertainty set Θ_2^2 is embedded into the polyhedral convex set $\Theta^2_{\infty\cap 1(\sqrt{2})}$,

$$\Theta^2_{\infty\cap 1(\sqrt{2})} := \left\{ \theta_{\mathrm{S7}} \in \mathbb{R}^2 | \gamma_l := \frac{\theta_{S7,l} - \rho^{pN}_{\mathrm{S7}}}{\rho^{pR}_{\mathrm{S7}}}, \right.$$

$$\left. l=1,2, ||\gamma||_1 \leq \sqrt{2}, ||\gamma||_\infty \leq 1 \right\}.$$

This step is necessary to derive a linear partially robust model of type (RC_2), see Appendix A. The partially robust storage constraints and the corresponding non-negativity constraints associated with product S7 with respect to $\Theta^2_{\infty\cap 1(\sqrt{2})}$ follow the lines of Eqs. 14 and 15 with the choice of

$$\Gamma^A_{sn(n'-1)} = \Gamma^B_{sn(n'-1)} = \sqrt{2}, \quad s=\mathrm{S7}, 2 \leq n \leq 5, 2 \leq n' \leq n.$$

The RHS vector of Eq. 14 additionally depends on θ_4 accounting for uncertainty in the storage capacity for S7. The partially robust storage constraints for product S7 are given by

$$sti_{\mathrm{S7}} - x_{\mathrm{S7},1} - b_{\mathrm{Sep,U4},1} \leq 200 + 50\theta_4$$

$$sti_{\mathrm{S7}} - x_{\mathrm{S7},1} - b_{\mathrm{Sep,U4},1}$$

$$+ \sum_{2 \leq n' \leq n} \left(-x_{\mathrm{S7},n'} - b_{\mathrm{Sep,U4},n'} + 0.9 \sum_{j \in \{\mathrm{U1,U2}\}} b_{\mathrm{R3},j(n'-1)} \right.$$

$$\left. + \sqrt{2} z^A_{\mathrm{S7},n(n'-1)} + \sum_{j \in \{\mathrm{U1,U2}\}} p^A_{\mathrm{S7},n(n'-1),\mathrm{R3},j} \right) \leq 200 + 50\theta_4,$$

$$2 \leq n \leq 5$$

$$z^A_{\mathrm{S7},n(n'-1)} + p^A_{\mathrm{S7}n(n'-1),R3,j} \geq 0.1 b_{\mathrm{R3},j(n'-1)},$$

$$2 \leq n \leq 5, 2 \leq n' \leq n, j \in \{\mathrm{U1,U2}\}$$

$$z^A_{\mathrm{S7},n(n'-1)} \geq 0, p^A_{\mathrm{S7},n(n'-1),\mathrm{R3},j} \geq 0,$$

$$2 \leq n \leq 5, 2 \leq n' \leq n, j \in \{\mathrm{U1,U2}\}.$$

The partially robust duration constraints with respect to Θ^1_∞ are given by,

$$t^f_{\mathrm{R1,U1},n} \geq t^s_{\mathrm{mix,U1},n} + \tfrac{2}{3}(3+\theta_5) w_{\mathrm{R1,U1},n} + \tfrac{2}{150} 4 b_{\mathrm{R1,U1},n}, n \leq 5.$$

Solving Example 4 with five revealing and two nonrevealing parameters, the two-stage method identifies 31 critical regions with a maximum of three profiles stored in the envelope for any of the regions. The partially robust model is infeasible for part of the initial feasible set. A selection of the critical regions and the corresponding profits is given in Appendix C, Table C5.

Case B. In contrast to Case A, the uncertainty set Θ_2 is now approximated by the box constrained uncertainty set Θ^2_∞,

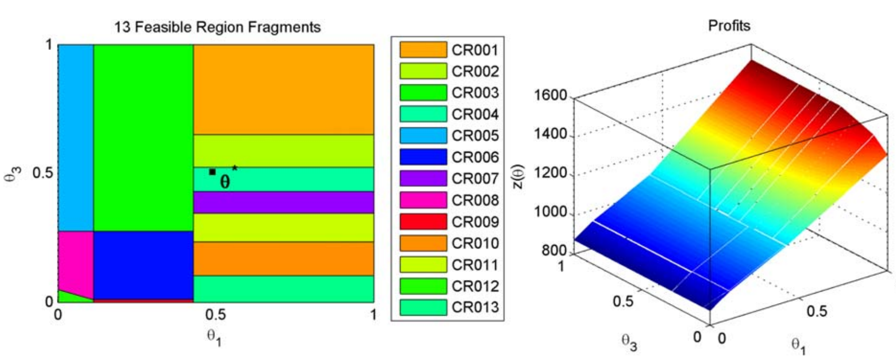

Figure 14. Critical regions and profit with two-stage method - Example 3, Case B, where $\theta_2=0$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

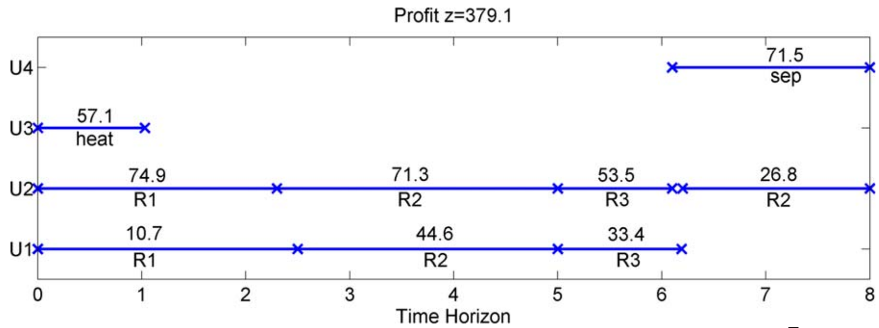

Figure 15. Gantt-chart for Example 4, Case A, with two-stage method at $\theta^*=(0,0,0,0,0)^T \in CR_4$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

$$\Theta_\infty^2 := \left\{ \theta_{S7} \in \mathbb{R}^2 | \gamma_l := \frac{\theta_{S7,l} - \rho_{S7}^{pN}}{\rho_{S7}^{pR}}, l=1,2, ||\gamma||_\infty \leq 1 \right\}.$$

Note that $\Theta_\infty^2 = \Theta_{\infty\cap 1(2)}^2$, as described in Appendix A, in this case. Hence, the partially robust model can be constructed using Eqs. 14 and 15 with

$$\Gamma_{sn(n'-1)}^A = \Gamma_{sn(n'-1)}^B = 2, \quad s=S7, 2 \leq n \leq 5, 2 \leq n' \leq n.$$

The two-stage method for Case B yields 34 critical regions with a multiplicity of three.

As the set Θ_∞ is less tight than $\Theta_{\infty\cap 1(\sqrt{2})}$ to enclose Θ_2, the approximate scheduling policy for Case B is expected to be more conservative than that for Case A. For example, at the parameter point $\theta^*=(0,0,0,0,0)^T \in \Theta_\infty^1$, the overall profit for Case B is $z=357.9$, whereas for Case A it is $z=379.1$. The approximate scheduling policies for Case A and Case B at θ^* are given in Figures 15 and 16, respectively.

The optimal policy for the nominal values, i.e. $\theta^N = (0,0,0,0,0)^T \in \Theta_\infty^1$ and $\theta_{S7}^N = (0.9, 0.9)^T \in \Theta_2^2$, is depicted in Figure 17. Here, the overall profit exceeds that for Case A and Case B at θ^*, which illustrates well the price for robustness in the proposed proactive scheduling approach.

Case C. We consider the case that the true values of the parameters $\theta_{i'}, i'=1,\ldots,5$, are not known to be available at the time of decision making. They belong the set Θ_∞^2 with

$$\Theta_\infty^2 := \left\{ \theta \in \mathbb{R}^5 | -1 \leq \theta_l \leq 1, l=1,\ldots,5 \right\}. \tag{17}$$

Furthermore, we assume that the production rates for S7 in U1 and U2, respectively, are attained at the nominal values, i.e. $\theta_{S7,1} = \theta_{S7,2} = \rho_{S7,R3}^N$. The partially robust scheduling model induced by Θ_∞^2 is then a deterministic problem. It is infeasible.

It immediately follows that the partially robust model with respect to the box constrained uncertainty set Θ_∞^2, Eq. 17, and the ellipsoidal uncertainty set Θ_2^2, Eq. 16, for the production rates is also infeasible. Note that as no multiparametric solver is required for the solution of the deterministic partially robust model with respect to Θ_∞^2 and Θ_2^2, it is not necessary to priorly embed Θ_2^2 into a polyhedral convex set, yielding a nonlinear robust counterpart problem.[19,42]

Example 5. This example is taken and altered from Wu and Ierapetritou.[41] The process consists of eight tasks to produce four final products from three feeds. There are nine intermediate products involved. The STN representation of Example 5 is given in Figure 18. There are six different units available. Task 1 is performed in unit U1, Task 2 in unit U4, Task 4 in unit U3, and Task 6 in unit U5. Task 3 and Task 7 share unit U2, and Task 5 and Task 8 share unit U6. The data for the process is given in Tables 7 and 8. It is assumed that the prices of all final products, as well as the cost to acquire the raw material may vary up to 25% from their nominal value. Demands for the final product and all processing times have 10% variability level. A total of 19 parameters are present in the scheduling formulation.

The time horizon is 18 h and 15 event points are employed in the scheduling model. The Gantt-chart for the optimal schedule of the nominal problem at $\theta_l^N = 0, l=1,\ldots,19$, is given in Figure C10 in Appendix C. For the nominal problem a profit of $z = 628330$ is generated by putting out 7178, 15344, 3855, and 13994 units, respectively, of final products S10, S11, S12, and S13.

Case A. The uncertainty sets are box constrained. Processing time and demand uncertainty are not known at the time

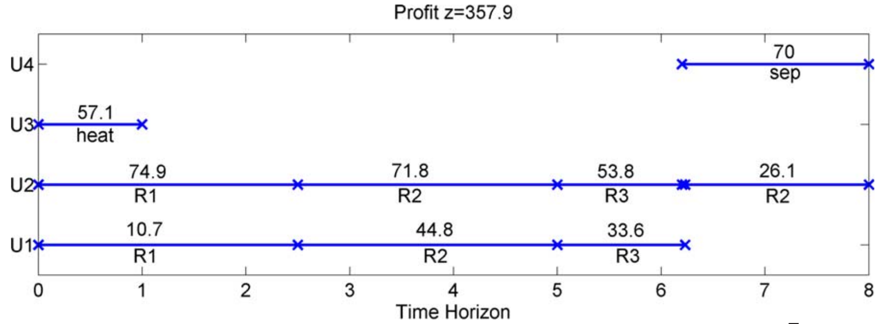

Figure 16. Gantt-chart for Example 4, Case B, with two-stage method at $\theta^*=(0,0,0,0,0)^T \in CR_4$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

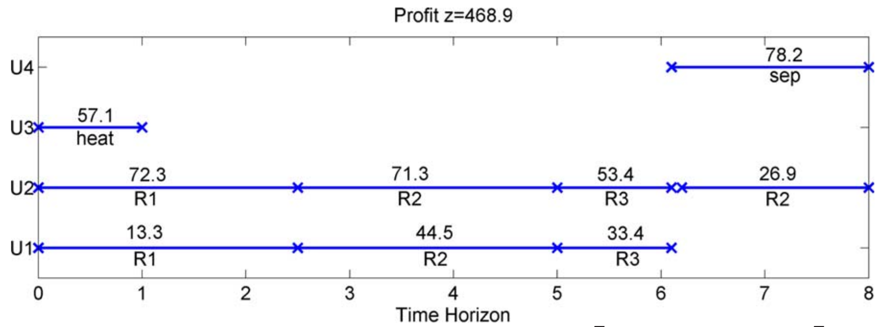

Figure 17. Gantt-chart for Example 4 at nominal values $\theta^N=(0,0,0,0,0)^T$ and $\theta^N_{S7}=(0.9,0.9)^T$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

of decision making. They belong to the class of nonrevealing parameters. Price and cost uncertainty are also box constrained, yet known to be available at the time of decision making. We have

$$\Theta^2_\infty := \{\theta_l \in \mathbb{R}, l=1,\ldots,12 | -1 \le \theta_l \le 1, l=1,\ldots,8, \\ -100 \le \theta_l \le 100, l=9,\ldots,12\},$$

and

$$\Theta^1_\infty := \{\theta_l \in \mathbb{R}, l=13,\ldots,19 | -4.5 \le \theta_{13} \le 4.5, -4.75 \\ \le \theta_{14} \le 4.75, 5 \le \theta_{15} \le 5, -5.25 \le \theta_{16} \le 5.25, \\ -1.25 \le \theta_l \le 1.25, l=17,\ldots,19\}.$$

The partially robust scheduling model contains robustified duration and demand constraints that are constructed analogously to Example 1, Case C. The constraints are obtained when variable and constant processing times, as well as the demands that must be met, are assigned the corresponding upper bounds from the set Θ^2_∞. Representatively, we give the corresponding duration constraints for Task 1 being processed in U1,

$$t^f_{\mathrm{T1,U1},n} \ge t^s_{\mathrm{T1,U1},n} + \tfrac{2}{3} 1.1 w_{\mathrm{T1,U1},n} + \tfrac{2}{3000} 1.1 b_{\mathrm{T1,U1},n}, \\ n=1,\ldots,15,$$

and the constraints for the demand that must be met for product S10,

$$\sum_{n=1}^{15} x_{\mathrm{S10},n} \ge 6600.$$

The revealing parameters from Θ^1_∞ appear in the objective function of the partially robust model.

The parametric scheduling policy obtained with the two-stage method yields 11 critical regions in Θ^1_∞. In Figure C11 in Appendix C, we depict the Gantt-chart for the proposed scheduling policy for the parameter point $\theta^*_l=0, l=13,\ldots,19$, which is contained in the first critical region. Function evaluation yields a profit of $z=600930$ and a production of 6770, 12994, 4025, and 14786, respectively, of the final products S10, S11, S12, and S13.

Compared to the nominal scheduling policy, the overall profit at θ^* is lower as a result of accounting for the worst case of duration and demand constraints in the partially robust model.

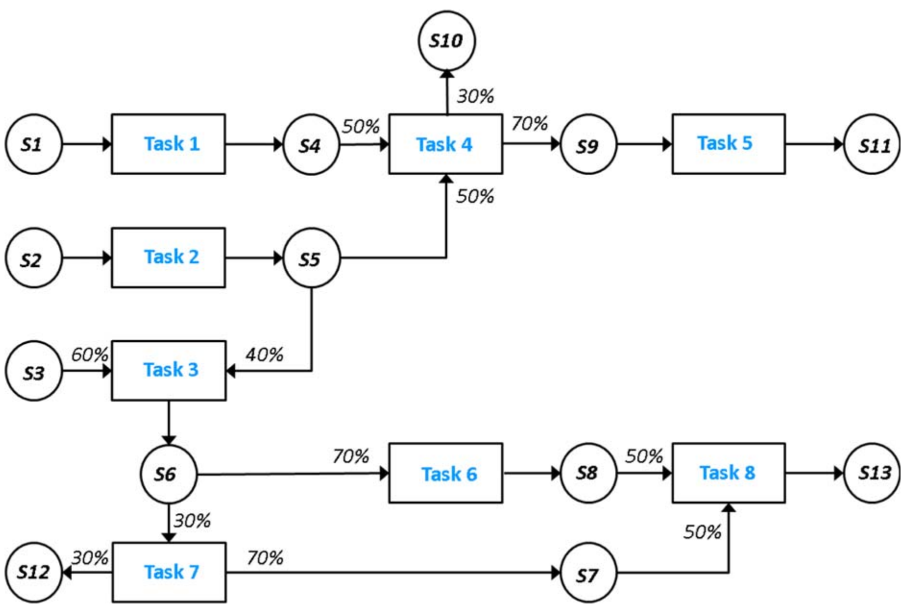

Figure 18. STN Representation of Example 5.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

Table 7. Data for Example 5.

Unit	Capacity	Task	Processing Time τ_{ij}
U1	1000	T1	$1+0.1\theta_1$
U2	2500	T3, T7	$1+0.1\theta_3$, $1+0.1\theta_7$
U3	3500	T4	$1+0.1\theta_4$
U4	1500	T2	$1+0.1\theta_2$
U5	1000	T6	$1+0.1\theta_6$
U6	4000	T5, T8	$1+0.1\theta_5$, $1+0.1\theta_8$

Case B. Here, only processing time is not known at the time of decision making. Demand, price and cost uncertainty are revealing parameters. We have

$$\Theta^2_\infty := \{\theta_l \in \mathbb{R}, l=1,\ldots,8 | -1 \le \theta_l \le 1, l=1,\ldots,8\},$$

and

$$\Theta^1_\infty := \{\theta_l \in \mathbb{R}, l=9,\ldots,19 | -100 \le \theta_l \le 100, l=9,\ldots,12, -4.5 \le \theta_{13} \le 4.5, -4.75 \le \theta_{14} \le 4.75, 5 \le \theta_{15} \le 5, -5.25 \le \theta_{16} \le 5.25, -1.25 \le \theta_l \le 1.25, l=17,\ldots,19\}.$$

The partially robust model accounts for the worst case with respect to processing time variability. Eleven parameters for uncertain demands, prices and costs remain present in the RHS of the constraints and in the objective function of the partially robust model. With the two-stage method, we obtain 40 critical regions in Θ^1_∞. For the realization $\theta^*_l=0, l=9,\ldots,19$, that belongs to the first critical region, a total profit of $z=552550$ is achieved. The amounts of the final products sold to the market are 6161, 14376, 3321, and 11500, respectively. The schedule is given in Figure C12 in Appendix C.

Note that the profit at θ^* is not improved compared to that for Case A, which has the more conservative setting. The limiting factor in this example is the solution of the mixed integer non-linear programming (MINLP) master problems in the decomposition algorithm employed in the two-stage method. In some iterations, the master problem reaches the limit of the execution time, reporting an integer feasible solution or without returning a solution at all. The latter case is treated as infeasible and the decomposition algorithm terminates in the corresponding region. These causes for suboptimality may be alleviated by further increasing the execution time of the solver. Moreover, it may become necessary to add the optimal policy of the worst case scenario when all parameters are assumed to be nonrevealing, Case C, to the envelope of parametric profiles. It ensures that for any given point from Θ^1_∞ function evaluation always yields an approximate scheduling policy that is less or equally conservative than that of the corresponding conventional robust scheduling model.

Case C. We consider the most restrictive case of all parameters being nonrevealing. The underlying uncertainty set is described by

$$\Theta^2_\infty := \{\theta_l \in \mathbb{R}, l=1,\ldots,19 | -1 \le \theta_l \le 1, l=1,\ldots,8, -100 \le \theta_l \le 100, l=9,\ldots,12, -4.5 \le \theta_{13} \le 4.5, -4.75 \le \theta_{14} \le 4.75, 5 \le \theta_{15} \le 5, -5.25 \le \theta_{16} \le 5.25, -1.25 \le \theta_l \le 1.25, l=17,\ldots,19\}.$$

The two-stage method features a partially robust model that is a purely deterministic problem. It accounts for the worst-case scenario of the duration constraints, the demand constraints, and the objective function, which is attained when θ_1 to θ_{12} and θ_{17} to θ_{19}, respectively, are at the corresponding upper bounds, and the parameters θ_{13} to θ_{16}, relating to price variability, are at the corresponding lower bounds according to Θ^2_∞.

The optimal scheduling policy for the worst-case scenario yields a profit of $z=371073$ by selling 6600, 12994, 3849, and 14792 units of S10, S11, S12, and S13. The Gantt-chart is depicted in Figure C13 in Appendix C. The profit provides a lower bound on those of Case A and B for every feasible parameter point from Θ^2_∞.

Case D. We consider the reverse scenario of Case B. Processing times are revealing parameters, whereas the data regarding demands, prices and costs are not expected to be available at the time of decision making. Both uncertainty sets are box constrained. We have

$$\Theta^1_\infty := \{\theta_l \in \mathbb{R}, l=1,\ldots,8 | -1 \le \theta_l \le 1, l=1,\ldots,8\},$$

and

$$\Theta^2_\infty := \{\theta_l \in \mathbb{R}, l=9,\ldots,19 | -100 \le \theta_l \le 100, l=9,\ldots,12, -4.5 \le \theta_{13} \le 4.5, -4.75 \le \theta_{14} \le 4.75, 5 \le \theta_{15} \le 5, -5.25 \le \theta_{16} \le 5.25, -1.25 \le \theta_l \le 1.25, l=17,\ldots,19\}.$$

In the partially robust scheduling model, Θ^1_∞ is responsible for the presence of parameters in the duration constraints. The duration constraints for Task 1 being processed in U1 now read as

$$t^f_{\mathrm{T1,U1},n} \ge t^s_{\mathrm{T1,U1},n} + \tfrac{2}{3}(1+0.1\theta_1)w_{\mathrm{T1,U1},n} + \tfrac{2}{3000}1.1 b_{\mathrm{T1,U1},n}, \quad n=1,\ldots,15.$$

The corresponding duration constraints for the remaining tasks are constructed analogously. Uncertainty induced by

Table 8. Data for Example 5.

State	Storage Capacity	Initial Amount	Initial Cost	Price	Demand
S1	–	–	$5+\theta_{17}$	0	0
S2	–	–	$5+\theta_{18}$	0	0
S3	–	–	$5+\theta_{19}$	0	0
S4,S5,S6	1000,1000,1500	0	0	0	0
S7,S8,S9	2000,0,3000	0	0	0	0
S10	–	0	0	$18+\theta_{13}$	$6000+6\theta_9$
S11	–	0	0	$19+\theta_{14}$	$8000+8\theta_{10}$
S12	–	0	0	$20+\theta_{15}$	$2000+2\theta_{11}$
S13	–	0	0	$21+\theta_{16}$	$8000+8\theta_{12}$

Table 9. Partially Robust Scheduling Formulations of Examples 1, 2, 3, 4, and 5.

	Constraints				Parameters	
	Inequalities	Equalities	Continuous Variables	Binary Variables	Original	In (*RC*)
Ex. 1, Case A	276	35	79	30	2	2
Ex. 1, Case B	276	35	79	30	2	1
Ex. 1, Case C	276	35	79	30	2	0
Ex. 2, Cases A,B,D,E	770	20	229	60	6	2
Ex. 2, Case C	680	20	199	60	2	2
Ex. 2, Case F	770	20	229	60	6	0
Ex. 3, Cases A,B	680	20	199	60	3	3
Ex. 4, Cases A,B	740	20	219	60	7	5
Ex. 4, Case C	680	20	199	60	5	0
Ex. 5, Case A	1934	90	433	210	19	7
Ex. 5, Case B	1934	90	433	210	19	11
Ex. 5, Case C	1934	90	433	210	19	0
Ex. 5, Case D	1935	90	433	210	19	2

the set Θ^2_∞ is accounted for accordingly to Case C. The partially robust scheduling model is an mp-MILP problem with LHS-uncertainty, which affect the entries of the constraint matrix affiliated with the binary variables. Eight parameters are involved.

For this case study, the two-stage method is not able to generate a parametric scheduling policy. At the solution stage, after the first iteration of the decomposition algorithm for the solution of the partially robust scheduling model more than 20,000 critical regions are established. The first iteration, which consists of solving the initial MINLP master problem and corresponding multiparametric linear programming (mp-LP) problem, requires a central processing unit (CPU) time of four days.

In order to derive a parametric scheduling policy, the number of revealing parameters is artificially reduced. We treat the processing times for Task 3 to Task 8 as nonrevealing data. The uncertainty sets are then described by

$$\Theta^1_\infty :=\{\theta_l \in \mathbb{R}, l=1,2 | -1 \le \theta_l \le 1, l=1,2\},$$

and

$$\begin{aligned}\Theta^2_\infty :=\{&\theta_l \in \mathbb{R}, l=3,\dots,19 | -1 \le \theta_l \le 1, l=3,\dots,8,\\ &-100 \le \theta_l \le 100, l=9,\dots,12, -4.5 \le \theta_{13} \le 4.5,\\ &-4.75 \le \theta_{14} \le 4.75, 5 \le \theta_{15} \le 5, -5.25 \le \theta_{16}\\ &\le 5.25, -1.25 \le \theta_l \le 1.25, l=17,\dots,19\}.\end{aligned}$$

With respect to the revised uncertainty sets, solely θ_1 and θ_2 remain explicitly in the partially robust scheduling model. The model is immunized against processing time uncertainty for Task 3 to Task 8. Furthermore, the following cut is added to the partially robust scheduling formulation

$$\sum_{n=1}^{15}\left(13.5x_{S10,n}+14.25x_{S11,n}+15x_{S12,n}+15.75x_{S13,n}\right) - \sum_{s\in S1,S2,S3} 6.25sti_s \ge 371073. \tag{18}$$

The value on the RHS of Eq. 18 is the profit for the worst case oriented scenario, Case C.

The solution stage of the two-stage method is further modified. The mitigated decomposition algorithm, which is discussed in the remark in Appendix B, is embedded in the two-stage method. With the modified two-stage method, we generate a scheduling policy that is not necessarily the optimal solution of the corresponding partially robust scheduling formulation. However, in combination with Eq. 18 it is enforced that the obtained solution yields a profit that is at least as conservative as that of the worst case oriented scenario for every $\theta \in \Theta^1_\infty$ where the corresponding profit is finite.

For our example, the partition of the $\theta_1-\theta_2$ space and the profit with the modified two-stage method are depicted in Figure C15 in Appendix D. For the specific parameter point $\theta^*_i=0,\ i=1,2,\ \theta^* \in CR_{49}$ an overall profit $z=376580$ is achieved. The amounts of S10, S11, S12, and S13 sold to the market are 6600, 14130, 3875, and 14000 units, respectively. The Gantt-chart is depicted in Figure C14 in Appendix C.

Table 10. Computational Requirements of the Two-Stage Method for Examples 1, 2, 3, 4, and 5.

	Critical Regions	Multiplicity	# MINLP	# mp-LP	CPU (d:h:min:s)
Ex. 1, Case A	5	2	9	3	0:39
Ex. 1, Case B	4	2	8	3	0:33
Ex. 2, Case A	7	1	8	1	7:25
Ex. 2, Case B	6	1	7	1	3:55
Ex. 2, Case C	3	1	4	1	3:09
Ex. 2, Case D	7	1	8	1	9:33
Ex. 2, Case E	4	1	5	1	3:01
Ex. 3, Case A	11	1	12	1	9:29
Ex. 3, Case B	13	1	14	1	7:49
Ex. 4, Case A	31	3	48	14	1:02:31
Ex. 4, Case B	34	3	59	23	1:16:34
Ex. 5, Case A	11	1	12	1	7:15:13
Ex. 5, Case B	40	2	44	4	1:03:36:54
Ex. 5, Case D	414	1	11	8	5:38:17

Implementation and computational requirements

Table 9 provides an overview of all partially robust scheduling models from this section. The second to last row provides the number of original parameters in the scheduling formulation. The last row contains the revealing parameters that remain explicitly in the partially robust scheduling model after the approximation step has been performed.

The decomposition algorithm for the solution of mp-MILP problems has been implemented in Matlab with an interface to GAMS 23.3[43] for the usage of BARON 9.0.2[44] and an interface to Cplex 12.1.[45] It is running on a Linux workstation (Dual 4 Core Intel Xeon processor, 1.6 GHz, 4 GB RAM) with Matlab being single threaded. In BARON, the relative optimality gap was set to 1% for Examples 1–4 and to 10% for Example 5. The resource time limit in seconds was set to 100 for Examples 1–4 and increased to 2000 for Example 5. Table 10 stores the computational requirements for the solution of all multiparametric partially robust mp-MILP scheduling formulations. The multiplicity of the envelope denotes the maximum of all numbers of solutions stored in the envelope of parametric profiles for any of the critical region.

The nominal scheduling models for Examples 1–5 in Gams-file format are available at http://sites.google.com/site/imperialercmobile/current-research/mp-milp-collection. Furthermore, the Matlab MAT-files of all partially robust scheduling formulation considered, and the corresponding proactive scheduling policies as obtained with the two-stage method are also provided on the website.

Conclusions

We address short-term batch process scheduling problems contaminated with uncertainty in the data using a two-stage method that combines state-of-the-art robust optimization and multiparametric programming techniques. The proposed two-stage method is able to account for uncertainty in the MILP scheduling model, spanning through the entries of the objective function and constraints. It is independent of the scheduling formulation employed and of the types of uncertainty we have studied, provided that all uncertain events can be modeled through the introduction of parameters, yielding a general mp-MILP problem of type (P). We explore several sources of uncertainty that affect the scheduling of batch processes, originating from varying prices, demands, time horizon, and storage capacities, but also from uncertain processing times and conversion rates.

At the optimization stage, a distinction is made between data that are known to be available at the time of decision making and that remain uncertain throughout. Whereas nonrevealing parameters in the scheduling model are treated with robust optimization techniques, revealing uncertainty, on the other hand, is addressed via multiparametric programming. As disturbances in the entries of the constraint matrices are the most challenging types of uncertainty in mp-MILP, they are also addressed using robust optimization. The two-stage method propagates a partially robust scheduling model that remains a multiparametric programming problem. In the case studies, we demonstrate the applicability of the two-stage method and the impact that different uncertainty sets, which include box constrained, budget parameter regulated box constrained or ellipsoidal uncertainty sets, have in the construction of the partially robust scheduling model and, consequently, on the quality of the approximate scheduling policy.

The benefit of multiparametric programming as a tool for proactive scheduling is that the model is solved analytically. The parametric profiles are being stored in a look-up table. Once the true values of the parameters are known, the anticipated scheduling policy is readily obtained via function evaluation from the profiles stored in the look-up table. The need for repetitive online optimization for different parameter realizations is avoided. The parametric solution may provide significant analytical results and a valuable insight into the scheduling process.

Issues in multiparametric programming are the computational requirements to obtain the parametric profile. For larger sized problems the solution of MINLP master problems to global optimality in the decomposition algorithm may prove challenging. In comparison, online solution of a single MILP scheduling formulation for fixed parameter values at the decision stage is itself not problematic for the examples from the case studies. With the two-stage method, the number of overall critical regions may grow prohibitively when the problem size or the number of parameters in the scheduling formulation increases. The number of parameters in the partially robust scheduling model may always be further reduced when part of the revealing parameters are treated as nonrevealing ones. The two-stage method yields a less or equally conservative solution compared to conventional robust optimization.

Ongoing work focuses on embedding novel piecewise affine relaxations of bilinear terms in the constraints that involve revealing parameters into the approximation stage.[46–48] Moreover, the potential of multiparametric programming as a tool for efficient rescheduling, enabling fast responses to changes that occur during the production process, is further investigated with a particular focus on the integration of scheduling and control.[8,49]

Acknowledgment

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Notation

H = time horizon
I = index set of tasks
J = index set of units
S = index set of tasks
I_j = index set of tasks performed in unit j
I_s = index set of tasks processing state s
J_i = index set of units suitable for performing task i
N = event points
x_{sn} = amount of state s sold at event point n
st_{sn}, sti_s = amount of state s at event point n, initial amount of state s
b_{ijn} = batch size processed by task i in unit j at event point n
t^s_{ijn}, t^f_{ijn} = starting and finishing time of task i in unit j at event point n
w_{ijn} = activation status of task i in unit j at event point n
P_s = price of state s
C_s = cost of initially acquiring state s
R_s = demand of state s
ρ^c_{si}, ρ^p_{si} = proportion of state s consumed and produced by task i
$\mathrm{ST}^{\max}_s$ = maximum storage capacity of state s
$V^{\min}_{ij}, V^{\max}_{ij}$ = minimum and maximum capacity for performing task i in unit j
τ_{ij} = mean processing time of task i in unit j
α_{ij}, β_{ij} = constant and variable processing time of task i in unit j, $\alpha_{ij} := \frac{2}{3}\tau_{ij}, \beta_{ij} := \frac{2}{3}\frac{\tau_{ij}}{V^{\max}_{ij} - V^{\min}_{ij}}$
(P) = multiparametric MILP problem
(RC) = partially robust counterpart model of (P)

x = vector of continuous variables
y = vector of continuous variables
$\theta, \theta^1, \theta^2$ = parameter vectors
q = number of parameters
$\Theta, \Theta^1, \Theta^2$ = feasible set of all, all revealing, and all nonrevealing parameters
Θ_∞ = box constrained uncertainty set
Θ_2 = ellipsoidal uncertainty set
$\Theta_{\infty\cap 1(\sqrt{q})}$ = combined box constrained and interval uncertainty set
Θ_Γ = budget parameter regulated uncertainty set
Γ = budget parameter
$\Gamma_s, \Gamma^{A,B}_{sn(n'-1)}$ = budget parameters for uncertain production rates
S_Γ = index set with $|S_\Gamma|=\Gamma$
z, p^i = vectors of auxiliary continuous variables in (RC_Γ)
$z^{A,B}_{sn(n'-1)}, p^{A,B}_{sn(n'-1)ij}$ = auxiliary variables in (RC_Γ) for uncertain production rates

Literature Cited

1. Floudas CA, Lin X. Mixed integer linear programming in process scheduling: modeling, algorithms, and applications. *Ann Oper Res.* 2005;139(1):131–162.
2. Verderame PM, Elia JA, Li J, Floudas CA. Planning and scheduling under uncertainty: a review across multiple sectors. *Ind Eng Chem Res.* 2010;49(9):3993–4017.
3. Li Z, Ierapetritou MG. Process scheduling under uncertainty: review and challenges. *Comput Chem Eng.* 2008;32(4-5):715–727.
4. Floudas CA, Lin X. Continuous-time versus discrete-time approaches for scheduling of chemical processes: a review. *Comput Chem Eng.* 2004;28(11):2109–2129.
5. Grossmann IE. Enterprise-wide optimization: a new frontier in process systems engineering. *AIChE J.* 2005;51(7):1846–1857.
6. Méndez CA, Cerdá J, Grossmann IE, Harjunkoski I, Fahl M. State-of-the-art review of optimization methods for short-term scheduling of batch processes. *Comput Chem Eng.* 2006;30(6-7):913–946.
7. Maravelias CT. General framework and modeling approach classification for chemical production scheduling. *AIChE J.* 2012;58(6):1812–1828.
8. Engell S, Harjunkoski I. Optimal operation: scheduling, advanced control and their integration. *Comput Chem Eng.* 2012;47:121–133.
9. Pistikopoulos E. Uncertainty in process design and operations. *Comput Chem Eng.* 1995;19(1):553–563.
10. Bonfill A, Bagajewicz M, Espuña A, Puigjaner L. Risk management in the scheduling of batch plants under uncertain market demand. *Ind Eng Chem Res.* 2004;43(3):741–750.
11. Vin JP, Ierapetritou MG. Robust short-term scheduling of multiproduct batch plants under demand uncertainty. *Ind Eng Chem Res.* 2001;40(21):4543–4554.
12. Balasubramanian J, Grossmann IE. Approximation to multistage stochastic optimization in multiperiod batch plant scheduling under demand uncertainty. *Ind Eng Chem Res.* 2004;43(14):3695–3713.
13. Bonfill A, Espuña A, Puigjaner L. Addressing robustness in scheduling batch processes with uncertain operation times. *Ind Eng Chem Res.* 2005;44(5):1524–1534.
14. Bonfill A, Espuña A, Puigjaner L. Proactive approach to address the uncertainty in short-term scheduling. *Comput Chem Eng.* 2008;32(8):1689–1706.
15. Balasubramanian J, Grossmann I. A novel branch and bound algorithm for scheduling flowshop plants with uncertain processing times. *Comput Chem Eng.* 2002;26(1):41–57.
16. Lin X, Janak S, Floudas C. A new robust optimization approach for scheduling under uncertainty:—I. Bounded uncertainty. *Comput Chem Eng.* 2004;28(6):1069–1085.
17. Janak S, Janak S, Floudas C. A new robust optimization approach for scheduling under uncertainty: II. Uncertainty with known probability distribution. *Comput Chem Eng.* 2007;31(3):171–195.
18. Li Z, Ierapetritou MG. Robust optimization for process scheduling under uncertainty. *Ind Eng Chem Res.* 2008;47(12):4148–4157.
19. Li Z, Ding R, Floudas CA. A comparative theoretical and computational study on robust counterpart optimization: I. Robust linear optimization and robust mixed integer linear optimization. *Ind Eng Chem Res.* 2011;50(18):10567–10603.
20. Pistikopoulos EN. Perspectives in multiparametric programming and explicit model predictive control. *AIChE J.* 2009;55(8):1918–1925.
21. Wallace SW. Decision making under uncertainty: is sensitivity analysis of any use? *Oper Res.* 2000;48(1):20–25.
22. Ryu JH, Dua V, Pistikopoulos EN. Proactive scheduling under uncertainty: a parametric optimization approach. *Ind Eng Chem Res.* 2007;46(24):8044–8049.
23. Ryu JH, Pistikopoulos EN. A novel approach to scheduling of zero-wait batch processes under processing time variations. *Comput Chem Eng.* 2007;31(3):101–106.
24. Dua V, Bozinis A, Pistikopoulos EN. A multiparametric programming approach for mixed-integer quadratic engineering problems. *Comput Chem Eng.* 2002;26:715–733.
25. Li Z, Ierapetritou MG. Process scheduling under uncertainty using multiparametric programming. *AIChE J.* 2007;53(12):3183–3203.
26. Li Z, Ierapetritou MG. Reactive scheduling using parametric programming. *AIChE J.* 2008;54(10):2610–2623.
27. Ierapetritou MG, Floudas CA. Effective continuous-time formulation for short-term scheduling. 1. Multipurpose batch processes. *Ind Eng Chem Res.* 1998;37(11):4341–4359.
28. Ierapetritou MG, Floudas CA. Effective continuous-time formulation for short-term scheduling. 2. Continuous and semicontinuous processes. *Ind Eng Chem Res.* 1998;37(11):4360–4374.
29. Ierapetritou MG, Hene TS, Floudas CA. Effective continuous-time formulation for short-term scheduling. 3. Multiple intermediate due dates. *Ind Eng Chem Res.* 1999;38(9):3446–3461.
30. Gal T, Nedoma J. Multiparametric linear programming. *Math Program Stud.* 1972;18:406–422.
31. Gal T. Rim multiparametric linear programming. *Manag Sci.* 1975;21(5):567–575.
32. Pertsinidis A, Grossmann IE, McRae GJ. Parametric optimization of MILP programs and a framework for the parametric optimization of MINLPs. *Comput Chem Eng.* 1998;22(Suppl 1):205–212.
33. Acevedo J, Pistikopoulos EN. A parametric MINLP algorithm for process synthesis problems under uncertainty. *Ind Eng Chem Res.* 1996;35(1):147–158.
34. Faísca NP, Kosmidis VD, Rustem B, Pistikopoulos EN. Global optimization of multi-parametric MILP problems. *J Glob Optim.* 2009;45(1):131–151.
35. Jia Z, Ierapetritou MG. Uncertainty analysis on the righthand side for MILP problems. *AIChE J.* 2006;52(7):2486–2495.
36. Borrelli F, Bemporad A, Morari M. Geometric algorithm for multiparametric linear programming. *J Optim Theory Appl.* 2003;118(3):515–540.
37. Li Z, Ierapetritou MG. A new methodology for the general multiparametric mixed-integer linear programming (MILP) problems. *Ind Eng Chem Res.* 2007;46(15):5141–5151.
38. Mitsos A, Barton PI. Parametric mixed-integer 0–1 linear programming: the general case for a single parameter. *Eur J Oper Res.* 2009;194(3):663–686.
39. Wittmann-Hohlbein M, Pistikopoulos EN. On the global solution of multi-parametric mixed integer linear programming problems. *J Glob Optim.* 2012;1–23, doi:10.1007/s10898-012-9895-2.
40. Wittmann-Hohlbein M, Pistikopoulos EN. A two-stage method for the approximate solution of general multiparametric mixed-integer linear programming problems. *Ind Eng Chem Res.* 2012;51(23):8095–8107.
41. Wu D, Ierapetritou MG. Decomposition approaches for the efficient solution of short-term scheduling problems. *Comput Chem Eng.* 2003;27(8-9):1261–1276.
42. Ben-Tal A, Nemirovski A. Robust solutions of linear programming problems contaminated with uncertain data. *Math Program.* 2000;88(3):411–424.
43. GAMS. GAMS Development Corporation. Available at: http://www.gams.com, accessed in 2011.
44. Sahinidis N, Tawarmalani M. Gams/Baron 9.0.2: global optimization of mixed-integer nonlinear programs, Available at http://www.gams.com/dd/docs/solvers/baron.pdf, accessed in 2009.
45. IBM ILOG CPLEX. Available at: www.ibm.com/software/integration/optimization/cplex-optimizer/, accessed in 2011.
46. Wicaksono DS, Karimi IA. Piecewise MILP under- and overestimators for global optimization of bilinear programs. *AIChE J.* 2008;54(4):991–1008.
47. Gounaris CE, Misener R, Floudas CA. Computational comparison of piecewise linear relaxations for pooling problems. *Ind Eng Chem Res.* 2009;48(12):5742–5766.
48. Misener R, Thompson JP, Floudas CA. APOGEE: global optimization of standard, generalized, and extended pooling problems via linear and logarithmic partitioning schemes. *Comput Chem Eng.* 2011;35(5):876–892.

49. Subramanian K, Maravelias CT, Rawlings JB. A state-space model for chemical production scheduling. *Comput Chem Eng*. 2012;47:97–110.
50. Bertsimas D, Sim M. Robust discrete optimization and network flows. *Math Program* 2003;98:49–71.
51. Bertsimas D, Sim M. The price of robustness. *Oper Res*. 2004;52: 35–53.
52. Murty KG. *Linear and Combinatorial Programming*. John Wiley & Sons, Inc. New York, 1976.
53. Ben-Tal A, Nemirovski A. Robust solutions of uncertain linear programs. *Oper Res Lett*. 1999;25(1):1–13.
54. Ben-Tal A, Nemirovski AS. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2001.
55. Gal T. *Postoptimal Analyses, Parametric Programming, and Related Topics: Degeneracy*, Multicriteria Decision Making, Redundancy. Berlin: W. de Gruyter, 1995.

Appendix A: Uncertainty Set Dependent Partially Robust Counterpart Problems

The partially robust model (RC) is derived from (P) given knowledge about the underlying uncertainty set Θ. We focus on the construction of (RC) with respect to Θ^1. When nonrevealing parameters from Θ^2 are present in (P), (RC) is obtained analogously provided that (P) is transformed such that nonrevealing parameters appear only in the entries of the constraint matrices. If the objective function is affected by uncertainty, i.e. it may be written in the form

$$\min_{x,y}\ \left(c+H^1\theta^1+H^2\theta^2\right)^T x+\left(d+L^1\theta^1+L^2\theta^2\right)^T y$$

where $\theta^1\in\Theta^1$, $\theta^2\in\Theta^2$, an auxiliary variable $t\in\mathbb{R}$ is introduced to rewrite the objective as

$$\min_{x,y,t}\ \left(c+H^1\theta^1\right)^T x+\left(d+L^1\theta^1\right)^T y+t,$$

and the constraint

$$\left(H^2\theta^2\right)^T x+\left(L^2\theta^2\right)^T y-t\le 0$$

is added. Likewise, RHS-uncertainty of the form

$$\begin{aligned}&\left[a_i^1\left(\theta^1\right)\right]^T x+\left[e_i^1\left(\theta^1\right)\right]^T y+\left[a_i^2\left(\theta^2\right)\right]^T x+\left[e_i^2\left(\theta^2\right)\right]^T y\\&\quad\le b_i+\left[f_i^1\right]^T\theta^1+\left[f_i^2\right]^T\theta^2\end{aligned}$$

is replaced by the following constraints

$$\begin{aligned}&\left[a_i^1\left(\theta^1\right)\right]^T x+\left[e_i^1\left(\theta^1\right)\right]^T y+\left[a_i^2\left(\theta^2\right)\right]^T x+\left[e_i^2\left(\theta^2\right)\right]^T y\\&\quad-\left(\left[f_i^2\right]^T\theta^2\right)v_i\le b_i+\left[f_i^1\right]\theta^1\\&1\le v_i\le 1\end{aligned}$$

for $i=1,\ldots,m$, where $v_i\in\mathbb{R}$ is an additional optimization variable.

For the sake of brevity, in the construction of the uncertainty set dependent partially robust counterpart problems that follow, we assume that $\Theta^2=\emptyset$ holds for $\Theta:=\Theta^1\times\Theta^2$ in (P).

Box constrained uncertainty

The pair $(\bar{x},\bar{y}):=(\bar{x}(\theta),\bar{y}(\theta)),\theta\in\Theta$ is called a *partially robust feasible solution* of (P) if

$$\forall\gamma\in\Theta: A^N\bar{x}+\sum_{l=1}^{q}\gamma_l A^l\bar{x}+E(\theta)\bar{y}\le b+F\theta \qquad \text{(A1)}$$

for any $\theta\in\Theta$. Let us consider the set

$$\Theta_\infty:=\left\{\theta\in\mathbb{R}^q\,|\,\theta_l^{\min}\le\theta_l\le\theta_l^{\max},\ l=1,\ldots,q\right\}.$$

Every parameter is bounded with r_l denoting the range,

$$r_l:=\left(\theta_l^{\max}-\theta_l^{\min}\right)/2,$$

and θ_l^N denoting the nominal value of θ_l,

$$\theta_l^N:=\theta_l^{\max}-r_l,$$

for $l\in\{1,\ldots,q\}$. It follows from Eq. A1 that a partially robust feasible solution, $(\bar{x},\bar{y})$, of (P) with respect to Θ_∞ satisfies

$$\begin{aligned}&[a_i^N]^T\bar{x}+[e_i^N]^T\bar{y}+\sum_{l=1}^{q}\Bigg(\theta_l^N[a_i^l]^T\bar{x}+\max_{-r_l\le\gamma_l\le r_l}\gamma_l[a_i^l]^T\bar{x}\\&\quad+\theta_l[e_i^l]^T\bar{y}\Bigg)\le b_i+[f_i]^T\theta,\end{aligned} \qquad \text{(A2)}$$

for every constraint, $i=1,\ldots,m$, and any $\theta\in\Theta_\infty$. Eq. A2 aggregates those constraints of (P) that are most difficult to maintain with respect to instances of matrix A. We are in the position to represent Eq. A2 through linear constraints, yielding the partially robust model (RC_∞) induced by Θ_∞,

$$(RC_\infty)\begin{cases} r(\theta) & :=\min_{x,u^i,y}\left((c+H\theta)^T x+(d+L\theta)^T y\right)\\ \text{s.t.} & [a_i^N]^T x+[e_i^N]^T y+\sum_{l=1}^{q}(\theta_l^N[a_i^l]^T x+\theta_l[e_i^l]^T y)\\ & \quad+\sum_{l=1}^{q}r_l u_l^i\le b_i+[f_i]^T\theta,\ i=1,\ldots,m\\ & -u_l^i\le[a_i^l]^T x\le u_l^i,\ l=1,\ldots,q,\ i=1,\ldots,m\\ & x\in\mathbb{R}^n,\ y\in\{0,1\}^p\\ & u^i\in\mathbb{R}^q,\ i=1,\ldots,m\\ & \theta\in\Theta_\infty,\end{cases}$$

which features RHS- and OFC-uncertainty, and uncertainty in the constraint matrix E.

Box constrained uncertainty with an adjustable degree of conservatism of the solution

Data variability may be confined in a way that merely a subset of the parameters $\theta\in\Theta_\infty$ are likely to change at the same time. This observation leads to the construction of the partially robust counterpart of (P) with an adjustable degree of conservatism of the solution as propagated by Bertsimas and Sim for the deterministic case.[50,51] A budget parameter Γ, $0\le\Gamma\le q$ is introduced into the model. It ensures that any Γ parameters are allowed to vary from their nominal values and the remaining $q-\Gamma$ parameters attain nominal values. The budget parameter is fixed *a priori*.

Let K be the index set of all parameters with cardinality $|K|=q$. The subset $S_\Gamma\subseteq K$ is the index set of parameters such that $|S_\Gamma|=\Gamma$. The inducing uncertainty set is then described as follows,

$$\begin{aligned}\Theta_\Gamma:=\{&\theta\in\mathbb{R}^q\,|\,\exists S_\Gamma:\theta_l^{\min}\le\theta_l\le\theta_l^{\max},\forall l\in S_\Gamma,\\&\theta_l=\theta_l^N,\forall l\in K\backslash S_\Gamma\}.\end{aligned}$$

For the special cases $\Gamma=0$ and $\Gamma=q$, the uncertainty set reduces to $\Theta_\Gamma=\left\{\left(\theta_1^N,\ldots,\theta_q^N\right)^T\right\}$, and inflates to $\Theta_\Gamma=\Theta_\infty$, respectively. In our framework, it is sufficient that Γ is integral.

With Eq. A1, the constraints immunized against uncertainty in A read as

$$\left[a_i^N\right]^T x+\left[e_i^N\right]^T y+\max_{\theta\in\Theta_\Gamma}\sum_{l=1}^{q}\theta_l\left[a_i^l\right]^T x+\sum_{l=1}^{q}\theta_l\left[e_i^l\right]^T y\le b_i+\left[f_i\right]^T\theta$$

for $i=1,\ldots,m$ and $\theta\in\Theta_\Gamma$.

Following the argumentation in the work of Bertsimas and Sim,[50,51] it holds

$$\begin{aligned}
&\max_{\theta\in\Theta_\Gamma}\sum_{l=1}^{q}\theta_l\left[a_i^l\right]^T x\\
=\;&\sum_{l=1}^{q}\theta_l^N\left[a_i^l\right]^T x+\max_t\sum_{l=1}^{q}t_l r_l\left|\left[a_i^l\right]^T x\right|\\
&\text{s.t.}\quad \sum_{l=1}^{q}t_l\le\Gamma\\
&\qquad 0\le t_l\le 1,\ l=1,\ldots,q\\
&\qquad t\in\mathbb{R}^q\\
=\;&\sum_{l=1}^{q}\theta_l^N\left[a_i^l\right]^T x+\min_{z,p}\left(\Gamma z+\sum_{l=1}^{q}p_l\right)\\
&\text{s.t.}\quad z+p_l\ge r_l\left|\left[a_i^l\right]^T x\right|,\ l=1,\ldots,q\\
&\qquad z\ge 0,\ p_l\ge 0,\ l=1,\ldots,q\\
&\qquad z\in\mathbb{R},\ p\in\mathbb{R}^q.
\end{aligned}\tag{A3}$$

The first equality in Eq. A3 stems from the observation that the maximum is attained if any Γ parameters are at their worst values. The second equality is derived from LP duality theory.[52]

Incorporating Eq. A3, the partially robust counterpart (RC_Γ) induced by the set Θ_Γ is given by

$$(RC_\Gamma)\left\{\begin{array}{ll}
r(\theta) & :=\min\limits_{x,u^i,p^i,z,y,}\left((c+H\theta)^T x+(d+L\theta)^T y\right)\\
\text{s.t.} & \left[a_i^N\right]^T x+\left[e_i^N\right]^T y+\sum_{l=1}^{q}\left(\theta_l^N\left[a_i^l\right]^T x+\theta_l\left[e_i^l\right]^T y\right)\\
& +\sum_{l=1}^{q}p_l^i+\Gamma z^i\le b_i+[f_i]^T\theta,\ i=1,\ldots,m\\
& z^i+p_l^i\ge r^l u_l^i,\ l=1,\ldots,q,\ i=1,\ldots,m\\
& p_l^i\ge 0,\ l=1,\ldots,q,\ i=1,\ldots,m\\
& z^i\ge 0,\ i=1,\ldots,m\\
& -u_l^i\le\left[a_i^l\right]^T x\le u_l^i,\ l=1,\ldots,q,\ i=1,\ldots,m\\
& x\in\mathbb{R}^n,\ y\in\{0,1\}^p\\
& z\in\mathbb{R}^m,\ p^i\in\mathbb{R}^q,\ u^i\in\mathbb{R}^q,\ i=1,\ldots,m\\
& \theta\in\Theta_\infty.
\end{array}\right.$$

Note that (RC_Γ) is solved explicitly for all $\theta\in\Theta_\infty$, which is a superset of Θ_Γ, but has the property that it is a polyhedral convex set.

Ellipsoidal uncertainty

The ellipsoidal uncertainty set is described by

$$\Theta_2=\left\{\theta\in\mathbb{R}^q|\gamma_l:=\left(\theta_l-\theta_l^N\right)/r_l,\ l=1,\ldots,q,\ \|\gamma\|_2=\sqrt{\gamma^T\gamma}\le 1\right\}$$

where the nominal value and the range for every parameter is known. Traditionally, the deterministic robust counterpart with respect to Θ_2 is no longer a linear model.[53] Embedded in the partially robust model, it yields a multiparametric nonlinear problem that is computationally demanding to solve explicitly. It therefore becomes necessary to embed Θ_2 into a polyhedral convex uncertainty set. Ideally, the superset is expected to be as tight as possible.

It holds $\Theta_2\subseteq\Theta_\infty$, which is direct consequence of the norm equivalence $\|\gamma\|_\infty\le\|\gamma\|_2$. On the other hand, constructing the uncertainty set

$$\Theta_{1\left(\sqrt{q}\right)}:=\left\{\theta\in\mathbb{R}^q|\gamma_l:=\left(\theta_l-\theta_l^N\right)/r_l,\ l=1,\ldots,q,\ \|\gamma\|_1=\sum_{l=1}^{q}|\gamma_l|\le\sqrt{q}\right\}$$

it follows from $\frac{1}{\sqrt{q}}\|\gamma\|_1\le\|\gamma\|_2$ that

$$\Theta_2\subseteq\Theta_\infty\cap\Theta_{1\left(\sqrt{q}\right)}=:\Theta_{\infty\cap 1\left(\sqrt{q}\right)}.$$

The set $\Theta_{\infty\cap 1\left(\sqrt{q}\right)}$ is defined as combined interval/box constrained and polyhedral uncertainty set in the work by Li et al.[19]

In the next step, the partially robust counterpart of (P) with respect to $\Theta_{\infty\cap 1\left(\sqrt{q}\right)}$ is derived. Let

$$K_\infty:=\left\{(y,t)\in\mathbb{R}^q\times\mathbb{R}|\|y\|_\infty\le t\right\}$$

and

$$K_1:=\left\{(y,t)\in\mathbb{R}^q\times\mathbb{R}|\|y\|_1\le t\right\}$$

denote the convex cones that induce a partial ordering in $\mathbb{R}^{q+1}$. The cones K_∞ and K_1 are dual to each other. We define $I\in\mathbb{R}^{q\times q}$ to be the identify matrix, and $\mathbf{0}\in\mathbb{R}^q$ the vector whose entries are all zero. It holds

$$\begin{aligned}
&\max_{\theta\in\Theta_{\infty\cap 1(\sqrt{q})}}\sum_{l=1}^{q}\theta_l[a_i^l]^T x\\
&=\sum_{l=1}^{q}\theta_l^N[a_i^l]^T x+\max_\gamma\left(\sum_{l=1}^{q}\gamma_l r_l[a_i^l]^T x\right)\\
&\text{s.t.}\begin{pmatrix} I\\ \mathbf{0}^T\end{pmatrix}\gamma+\begin{pmatrix}\mathbf{0}\\ 1\end{pmatrix}\in K_\infty\\
&\qquad\begin{pmatrix} I\\ \mathbf{0}^T\end{pmatrix}\gamma+\begin{pmatrix}\mathbf{0}\\ \sqrt{q}\end{pmatrix}\in K_1\\
&=\sum_{l=1}^{q}\theta_l^N[a_i^l]^T x+\min_{p,s,t,z}(s+\sqrt{q}z)\\
&\text{s.t.}\quad p_l+t_l=r_l[a_i^l]^T x,\ l=1,\ldots,q\\
&\qquad\sum_{l=1}^{q}|p_l|\le s,\ \max_l\{|t_l|\}\le z\\
&\qquad p\in\mathbb{R}^q,\ s\in\mathbb{R},\ t\in\mathbb{R}^q,\ z\in\mathbb{R}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{q} \theta_l^N [a_i^l]^T x + \min_{p,z} \left(\sum_{l=1}^{q} |p_l| + \sqrt{q} z \right) \\
&\text{s.t.} \quad \max_l \{ |r_l [a_i^l]^T x - p_l| \} \le z \\
&\qquad p \in \mathbb{R}^q, z \in \mathbb{R}
\end{aligned}
\tag{A4}
$$

where the second equality stems from conic duality theory.[54] Following the argumentation from the work by Li et al.,[19] the optimal solution of the last minimization problem of Eq. A4 is attained when $|p_l| \le |r_l [a_i^l]^T x|$, $l=1, \ldots, q$, yielding the transformation

$$
\begin{aligned}
= \quad & \sum_{l=1}^{q} \theta_l^N [a_i^l]^T x + \min_{p,z} \left(\sum_{l=1}^{q} |p_l| + \sqrt{q} z \right) \\
& \text{s.t.} \quad \max_l \left\{ |r_l [a_i^l]^T x - p_l| \right\} \le z \\
& \qquad p \in \mathbb{R}^q, z \in \mathbb{R} \\
= \quad & \sum_{l=1}^{q} \theta_l^N [a_i^l]^T x + \min_{p,z} \left(\sum_{l=1}^{q} |p_l| + \sqrt{q} z \right) \\
& \text{s.t.} \quad |r_l [a_i^l]^T x| - |p_l| \le z, l=1, \ldots, q \\
& \qquad z \ge 0 \\
& \qquad p \in \mathbb{R}^q, z \in \mathbb{R} \\
= \quad & \sum_{l=1}^{q} \theta_l^N [a_i^l]^T x + \min_{p,z} \left(\sum_{l=1}^{q} p_l + \sqrt{q} z \right) \\
& \text{s.t.} \quad z + p_l \ge r_l |[a_i^l]^T x|, l=1, \ldots, q \\
& \qquad z \ge 0, p \ge 0 \\
& \qquad p \in \mathbb{R}^q, z \in \mathbb{R}.
\end{aligned}
$$

The latter relation is incorporated in the construction of the partially robust counterpart. The partially robust counterpart of (P) with respect to Θ_2 reads as follows

$$
(RC_2) \begin{cases}
r(\theta) & := \min_{x, u^i, p^i, z, y} \left((c + H\theta)^T x + (d + L\theta)^T y \right) \\
\text{s.t.} & [a_i^N]^T x + [e_i^N]^T y + \sum_{l=1}^{q} \left(\theta_l^N [a_i^l]^T x + \theta_l [e_i^l]^T y \right) \\
& \quad + \sum_{l=1}^{q} p_l^i + \sqrt{q} z^i \le b_i + [f_i]^T \theta, i=1, \ldots, m \\
& z^i + p_l^i \ge r^l u_l^i, l=1, \ldots, q, i=1, \ldots, m \\
& p_l^i \ge 0, l=1, \ldots, q, i=1, \ldots, m \\
& z^i \ge 0, i=1, \ldots, m \\
& -u_l^i \le [a_i^l]^T x \le u_l^i, l=1, \ldots, q, i=1, \ldots, m \\
& x \in \mathbb{R}^n, y \in \{0,1\}^p \\
& z \in \mathbb{R}^m, p^i \in \mathbb{R}^q, u^i \in \mathbb{R}^q, i=1, \ldots, m \\
& \theta \in \Theta_{\infty \cap 1(\sqrt{q})}.
\end{cases}
$$

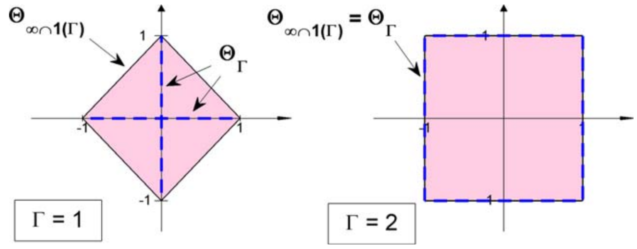

Figure A1. Uncertainty sets Θ_Γ and $\Theta_{\infty\cap 1(\Gamma)}$ as the areas enclosed by the dotted line and solid line, respectively, for the choice of $\Gamma=1$ and $\Gamma=2$ in $\mathbb{R}^2$.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com].

We denote by (RC_2) the partially robust mp-MILP model of (P) given the uncertainty set Θ_2. Note that (RC_2) is, in fact, the partially robust counterpart induced by the larger uncertainty set $\Theta_{\infty\cap 1(\sqrt{q})}$. For ellipsoidal uncertainty, the price of a partially robust counterpart of (P) that is an mp-MILP model, is higher than necessary protection level against uncertainty in the entries of the constraint matrix A. A feasible solution of (RC_2) is, consequently, a feasible solution of (P) for every realization of parameters from the set Θ_2.

Proposition 1. It holds $\Theta_\Gamma \subseteq \Theta_{\infty\cap 1(\Gamma)}$.

Proof 1. Let $\theta^* \in \Theta_\Gamma$, then there is an index set S_Γ^* with $\theta_l^* = \theta_l^N$ for all $l \in K \; S_\Gamma^*$ and

$$
\sum_{l \in K} \left| \frac{\theta_l^* - \theta_l^N}{r_l} \right| \le \sum_{l \in S_\Gamma^*} \left| \frac{(\theta_l^N + r_l) - \theta_l^N}{r_l} \right| \le \Gamma.
$$

Thus, $\theta^* \in \Theta_{1(\Gamma)}$. By definition $\theta^* \in \Theta_\infty$, which completes the proof.

Remark 2. *We observe that $(RC_\Gamma) = (RC_{\infty\cap 1(\Gamma)})$ for every $\theta \in \Theta_{\infty\cap 1(\Gamma)}$, where $(RC_{\infty\cap 1(\Gamma)})$ is constructed along the lines of (RC_2). Therefore, Problem (RC_Γ) may likewise be viewed as the corresponding partially robust counterpart for all parameters from the uncertainty set $\Theta_{\infty\cap 1(\Gamma)}$. The relation between Θ_Γ and $\Theta_{\infty\cap 1(\Gamma)}$ in $\mathbb{R}^2$ is depicted in Figure* A1.

Appendix B: A Decomposition Algorithm for the Solution of mp-MILP Problems

We briefly revisit the steps of a decomposition algorithm suitable for the solution of (RC), an mp-MILP problem of type (P) with A being parameter independent, which is based on the work of Dua et al.,[24] Faísca et al.[34] and Wittmann-Hohlbein and Pistikopoulos.[40] Problem (RC) is decomposed into a deterministic master problem and a mp-LP problem. The master problem (M) is derived from (RC) by treating the parameters as optimization variables. In general, it is an MINLP problem that needs to be solved to global optimality. The optimal integer node of (M) is fixed in (RC), which then yields an mp-LP subproblem (S). We assume that (S) is bounded from below for every feasible parameter point. The critical regions of (S), each a subset of the feasible set of parameters in which a particular basis remains optimal, are defined by the LP optimality conditions.[55] The union of all critical regions is convex.[31] Regions, where (S) is infeasible, are described by nonoverlapping, polyhedral convex sets and added to the list of critical regions with corresponding objective value of ∞.

Between every master and subproblem iteration performed over each incumbent partition of the parameter space, the master problem is updated. Integer cuts are introduced into the formulation of (M) in order to exclude previously visited integer solutions and parametric cuts ensure that only integer nodes that are optimal for (RC) for a certain realization of the parameters are considered. The cuts are given by

$$\sum_{l \in L^k := \left\{ l | y_l^k = 1 \right\}} y_l - \sum_{l \in \left\{ l | y_l^k = 0 \right\}} y_l \le |L^k| - 1, \quad k = 1, \ldots, \bar{K}$$

and

$$(c + H\theta)^T x + (d + L\theta)^T y \le z_k(\theta) - \epsilon, \quad k = 1, \ldots, \bar{K},$$

$\epsilon \ge 0$. By default we set $\epsilon = 1$. $\bar{K}$ denotes the number of previously identified integer solutions in the incumbent region, and $z_k(\theta)$ denotes the optimal objective value of (RC) at the integer node related to index k. The algorithm then terminates in a region where the master problem is infeasible.

A feature of the algorithmic procedure is that it operates on polyhedral convex partitions of the parameter space. Upon termination, for each critical region identified, a set of candidate solutions stored in the envelope of parametric profiles rather than solely the optimal solution of (*RC*) may be returned. To summarize, the final critical regions of (RC) obtained by the decomposition algorithm are polyhedral convex and the solutions stored in the envelope are affine functions in each region.

Remark 3. *The decomposition algorithm may be mitigated to identify a feasible solution, but not necessarily the optimal solution of* (*RC*). *The following adjustments are made. In the course of iteration, an updated MINLP master problem is only solved in those regions where the incumbent mp-LP problem is infeasible. Once a feasible solution of* (*RC*), *obtained from solving the incumbent mp-LP problem, is established in a particular region, this region is not explored further. The mitigated decomposition algorithm terminates in a region when either the incumbent mp-LP problem has an optimal solution or the updated MINLP master problem is infeasible. The latter indicates that problem* (*RC*) *is infeasible in this region. Compared with the decomposition algorithm, the mitigated decomposition algorithm requires the solution of fewer MINLP master problems and possibly fewer mp-LP subproblems. Upon termination, at most one solution is stored in the envelope of parametric profiles for each critical region.*
In Example 5, Case D, the mitigated decomposition algorithm is embedded in the modified two-stage method.

Appendix C: Supplement Material for Examples 2, 3, 4, and 5

Table C1. Partially Robust Profits—Example 2, Case A.

	Critical Region	Profits
CR_1	$\{\theta_1 - 0.15\theta_2 \le 0.99, -\theta_1 + 0.06\theta_2 \le -0.72, \theta_1 \le 1, 0 \le \theta_2 \le 1\}$	$512.08\theta_1 + 367.87\theta_2 + 718.85$
CR_2	$\{-\theta_1 + 0.15\theta_2 \le -0.99, \theta_1 \le 1, 0 \le \theta_2\}$	$512.08\theta_1 + 367.87\theta_2 + 718.85$
CR_3	$\{0.5 \le \theta_1, \theta_1 + 0.06\theta_2 \le 0.72, 0 \le \theta_2 \le 1\}$	$429.9\theta_2 + 372.94\theta_2 + 778.51$
CR_4	$\{0.4 \le \theta_1 \le 0.5, 0 \le \theta_2 \le 1\}$	$415\theta_1 + 372.94\theta_2 + 786.31$
CR_5	$\{0.2 \le \theta_1 \le 0.4, 0 \le \theta_2 \le 1\}$	$415\theta_1 + 372.94\theta_2 + 786.31$
CR_6	$\{0.17 \le \theta_1 \le 0.2, 0 \le \theta_2 \le 1\}$	$-400\theta_1^2 + 715.38\theta_1 + 372.94\theta_2 + 740.54$
CR_7	$\{0 \le \theta_1 \le 0.17, 0 \le \theta_2 \le 1\}$	$-400\theta_1^2 + 24.52\theta_1\theta_2 + 788.95\theta_1 + 368.62\theta_2 + 727.58$

Table C2. Partially Robust Profits—Example 2, Case B.

	Critical Region	Profits
CR_1	$\{0.66 \le \theta_1 \le 1, 0 \le \theta_2 \le 1\}$	$491.8\theta_1 + 351\theta_2 + 627.63$
CR_2	$\{0.4 \le \theta_1 \le 0.66, 0 \le \theta_2 \le 1\}$	$390\theta_1 + 351\theta_2 + 695.5$
CR_3	$\{0.27 \le \theta_1 \le 0.4, 0 \le \theta_2 \le 1\}$	$390\theta_1 + 351\theta_2 + 695.5$
CR_4	$\{0.22 \le \theta_1 \le 0.27, 0 \le \theta_2 \le 1\}$	$-400\theta_1^2 + 766.67\theta_1 + 351\theta_2 + 622.17$
CR_5	$\{0.02 \le \theta_1 \le 0.22, 0 \le \theta_2 \le 1\}$	$-400\theta_1^2 + 766.67\theta_1 + 351\theta_2 + 622.17$
CR_6	$\{0.17 \le \theta_1 \le 0.2, 0 \le \theta_2 \le 1\}$	$-400\theta_1^2 + 122\theta_1\theta_2 + 1100.3\theta_1 + 348.49\theta_2 + 615.33$

Table C3. Selected Partially Robust Profits—Example 3, Case A.

	Critical Region	Profits
CR_1	$\{0.4 \le \theta_1 \le 1, 0 \le \theta_2 \le 1, 0.5 \le \theta_3 \le 1\}$	$758.33\theta_1 + 394.87\theta_2 + 762.12$
CR_4	$\{0.4 \le \theta_1 \le 1, 0 \le \theta_2 \le 1, 0.2 \le \theta_3 \le 0.34\}$	$427.63\theta_1\theta_3 + 610.44\theta_1 + 394.88\theta_2 - 183.27\theta_3 + 825.51$
CR_8	$\{-0.3\theta_1 - \theta_3 \le -0.05, 0 \le \theta_1 \le 0.1, 0 \le \theta_2 \le 1, \theta_3 \le 0.27\}$	$-400\theta_1^2 + 671.43\theta_1 + 394.88\theta_2 + 872.84$
CR_{10}	$\{0.4 \le \theta_1 \le 1, 0 \le \theta_2 \le 1, 0 \le \theta_3 \le 0.1\}$	$1140.4\theta_1\theta_3 + 536.8\theta_1 + 394.88\theta_2 - 488.72\theta_3 + 857.07$

Table C4. Selected Partially Robust Profits—Example 3, Case B.

	Critical Region	Profits
CR_1	$\{0.4 \le \theta_1 \le 1, \theta_2 = 0, 0.65 \le \theta_3 \le 1\}$	$758.33\theta_1 + 762.12$
CR_3	$\{0.1 \le \theta_1 \le 0.4, \theta_2 = 0, 0.2 \le \theta_3 \le 1\}$	$455\theta_1 + 892.12$
CR_8	$\{0 \le \theta_1 \le 0.1, \theta_2 = 0, \theta_3 \le 0.27, -0.3\theta_1 - \theta_3 \le -0.05\}$	$-400\theta_1^2 + 671.43\theta_1 + 872.84$
CR_{10}	$\{0.4 \le \theta_1 \le 1, \theta_2 = 0, 0.1 \le \theta_3 \le 0.23\}$	$427.63\theta_1\theta_3 + 610.44\theta_1 - 183.27\theta_3 + 825.51$

Table C5. Selected Partially Robust Profits—Example 4, Case A

	Critical Region	Envelope of Profits
CR_1	$\{-1 \le \theta_1 \le 1, 0.2 \le \theta_2 \le 1, -1 \le \theta_3 \le 1, -1 \le \theta_4 \le 1, -1 \le \theta_5 \le -0.9\}$	$227.5\theta_1+260\theta_2-9.2\theta_5+521.08$
CR_3	$\{-1 \le \theta_1 \le 1, -\theta_2+0.1\theta_5 \le -0.33, -0.875\theta_1-\theta_2 \le 0.5, \theta_2 \le 1, -1 \le \theta_3 \le 1, -1 \le \theta_4 \le 1, -0.9 \le \theta_5 \le 0.33\}$	$-10.7\theta_1\theta_5-12.3\theta_2\theta_5+216.9\theta_1+247.8\theta_2-25\theta_5+505.44$
CR_{10}	$\{-1 \le \theta_1 \le 1, -1 \le \theta_3 \le 1, -1 \le \theta_4 \le 1, \theta_2-0.24\theta_5 \le -0.2, \theta_2-0.3\theta_5 \le -0.19, -\theta_2+0.26\theta_5 \le 0.2, \theta_5 \le 1\}$	$-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+4725.7\theta_2-1179.6\theta_5+1203.3$
CR_{13}	$\{-1 \le \theta_1 \le 1, -1 \le \theta_3 \le 1, -1 \le \theta_4 \le 1, -0.8\theta_1-\theta_2 \le -0.04, -\theta_2+0.6\theta_5 \le 0.08, 0.8\theta_1+\theta_2 \le 1.75, 0.77 \le \theta_5 \le 1\}$	$z^1=208.51\theta_1+238.3\theta_2+485.92$
		$z^2=-64.2\theta_1\theta_5-73.4\theta_2\theta_5+258.53\theta_1+295.47\theta_2-149.72+602.48$
CR_{27}	$\{-1 \le \theta_1 \le 1, 0.2 \le \theta_2 \le 0.4, -1 \le \theta_3 \le 1, -1 \le \theta_4 \le 1, -0.8\theta_2-\theta_5 \le -1.13, -\theta_2+0.74\theta_5 \le 0.48\}$	$z^1=-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+723.83\theta_2+304.15$
		$z^2=-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+1591.9\theta_2-55.9$
		$z^3=-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+485.71\theta_2-278.22\theta_5+619.55$
CR_{31}	$\{-1 \le \theta_1 \le 1, 0.16 \le \theta_2 \le 0.2, -1 \le \theta_3 \le 1, -1 \le \theta_4 \le 1, -0.8\theta_2-\theta_5 \le -1.13, -1 \le \theta_5\}$	$z^1=-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+723.83\theta_2+304.15$
		$z^2=-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+1867.5\theta_2+54.65$
		$z^3=-100\theta_1\theta_2-114.29\theta_2^2+250\theta_1+485.71\theta_2-278.22\theta_5+619.55$

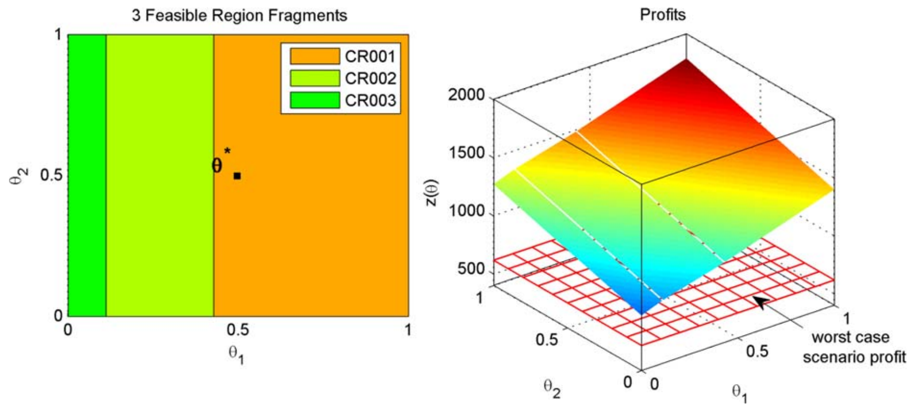

Figure C1. Critical regions and profit with two-stage method—Example 2, Case C.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

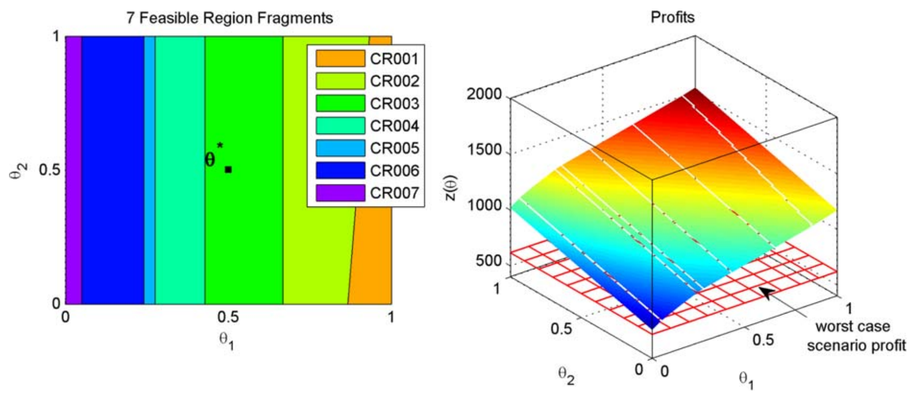

Figure C2. Critical regions and profit with two-stage method—Example 2, Case D.

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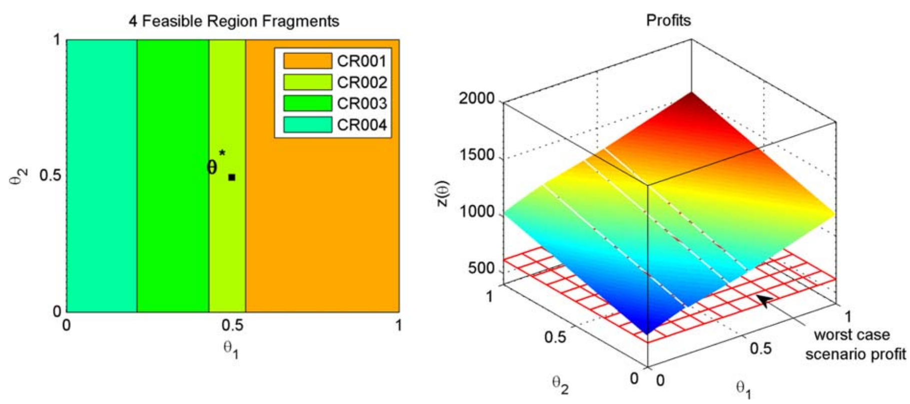

Figure C3. Critical regions and profit with two-stage method—Example 2, Case E.

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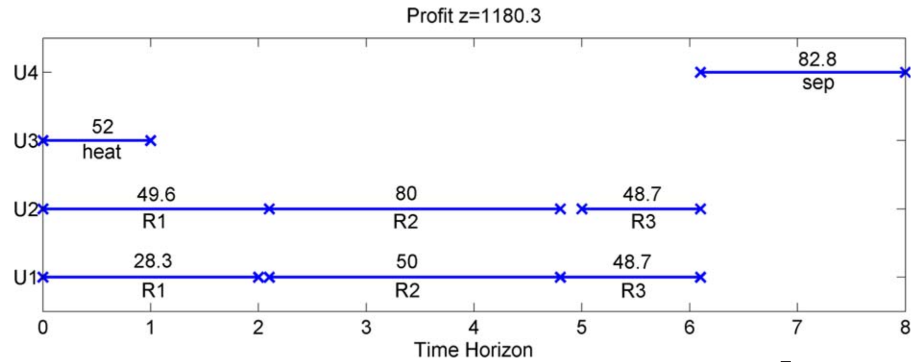

Figure C4. Gantt-chart for Example 2, Case A, with two-stage method at $\theta^* = (0.5, 0.5)^T \in CR_4$.

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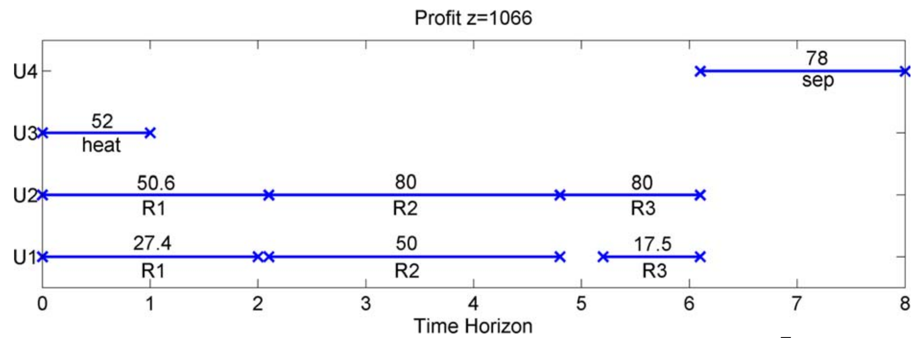

Figure C5. Gantt-chart for Example 2, Case B, with two-stage method at $\theta^* = (0.5, 0.5)^T \in CR_2$.

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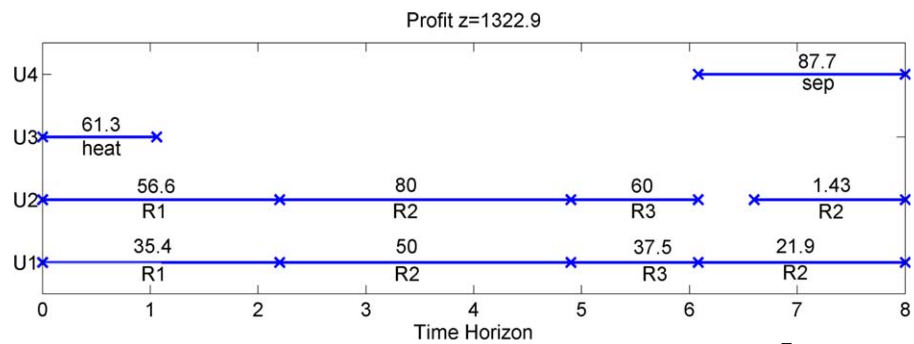

Figure C6. Gantt-chart for Example 2, Case C, with two-stage method at $\theta^* = (0.5, 0.5)^T \in CR_1$.

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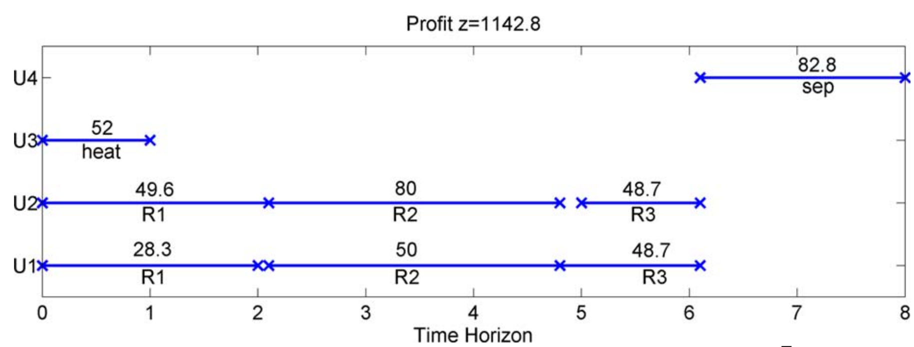

Figure C7. Gantt-chart for Example 2, Case D, with two-stage method at $\theta^* = (0.5, 0.5)^T \in CR_3$.

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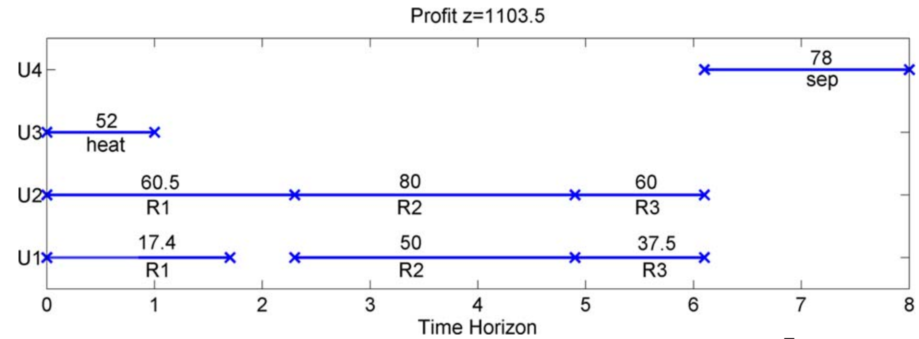

Figure C8. Gantt-chart for Example 2, Case E, with two-stage method at $\theta^{*}=(0.5,0.5)^{T}\in CR_2$

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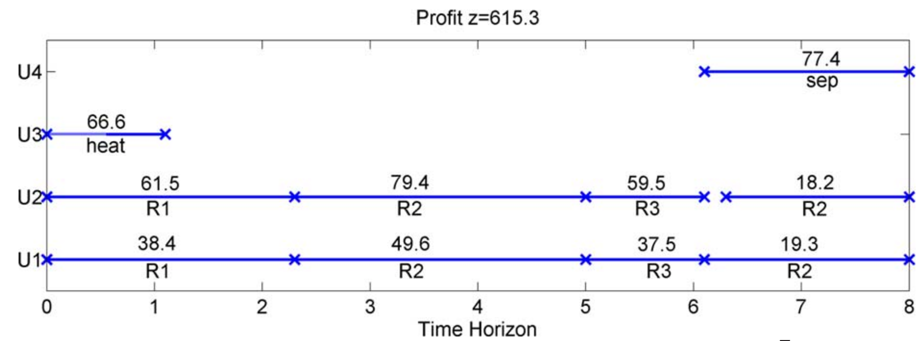

Figure C9. Gantt-chart for Example 2, Case F, with two-stage method at $\theta^{*}=(0.5,0.5)^{T}$.

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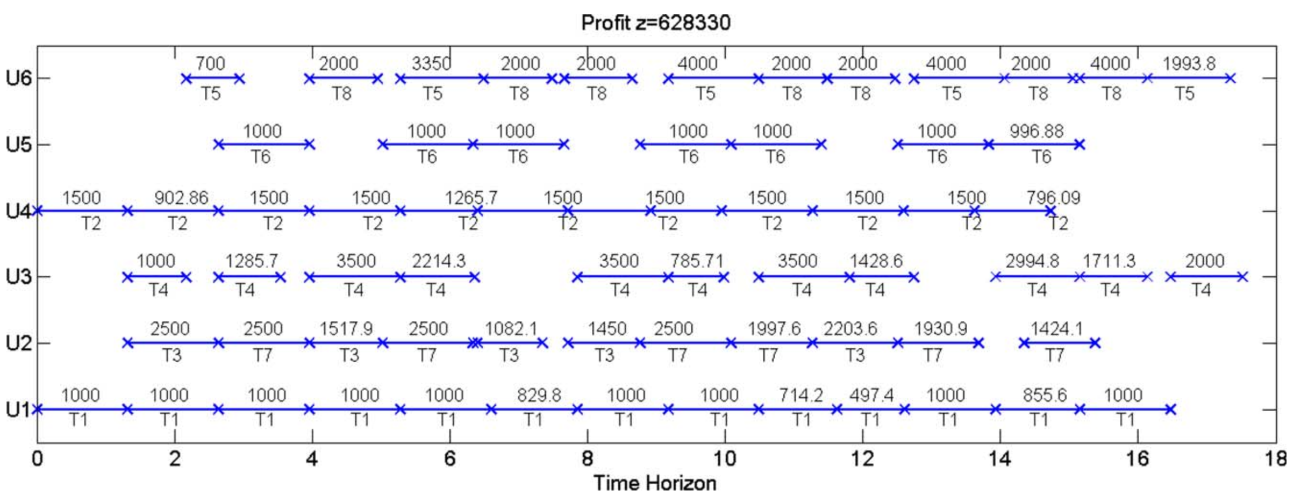

Figure C10. Gantt-chart for Example 5 at nominal value $\theta_l^N=0, l=1,\ldots,19$.

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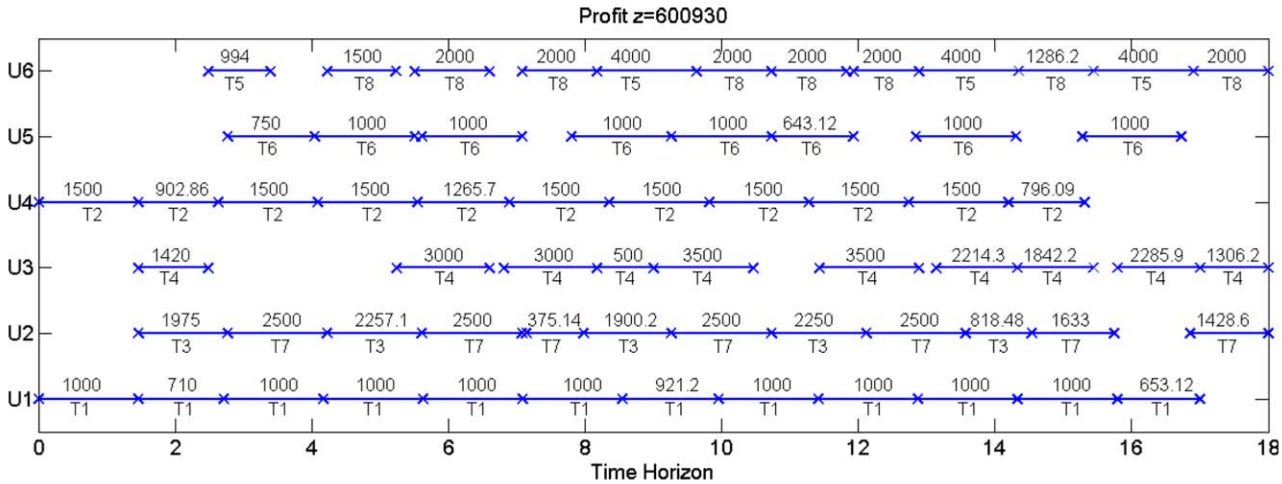

Figure C11. Gantt-chart for Example 5, Case A, with two-stage method at $\theta_l^*=0, l=13,...,19, \theta^* \in CR_1$.

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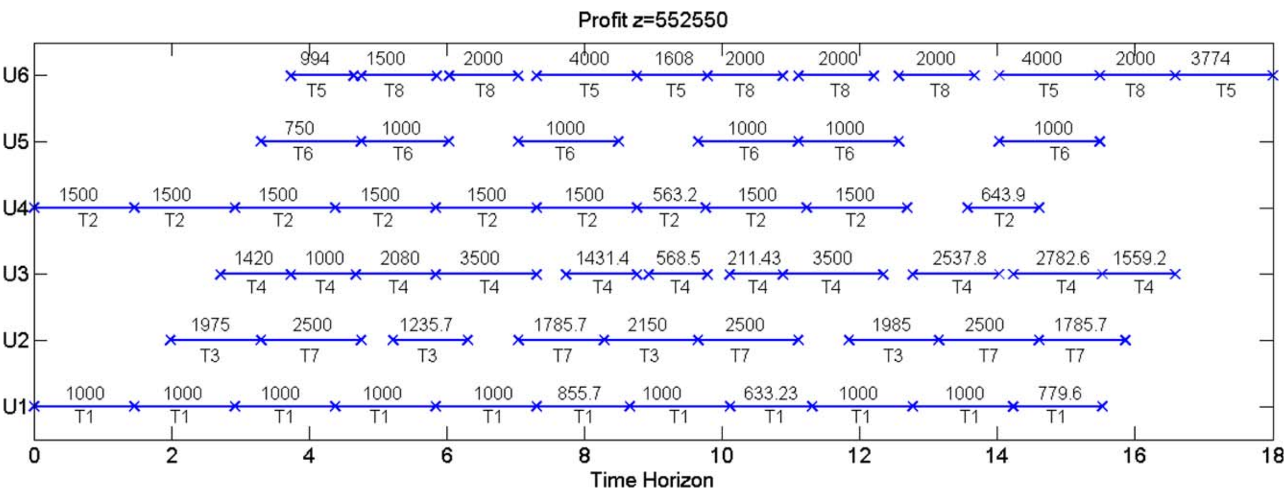

Figure C12. Gantt-chart for Example 5, Case B, with two-stage method at $\theta_l^*=0, l=9,...,19, \theta^* \in CR_1$.

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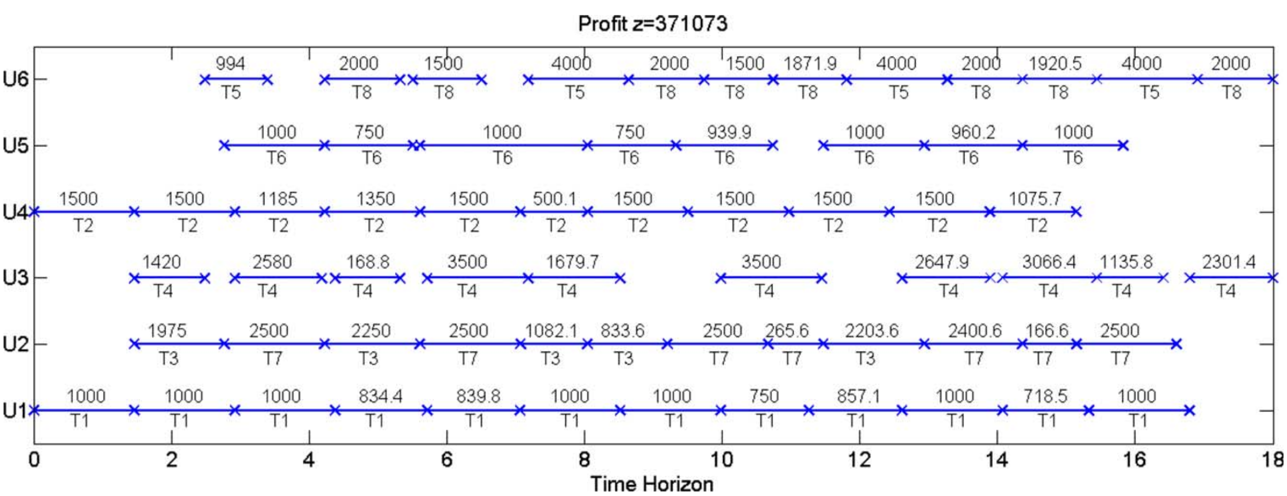

Figure C13. Gantt-chart for Example 5, Case C, with two-stage method.

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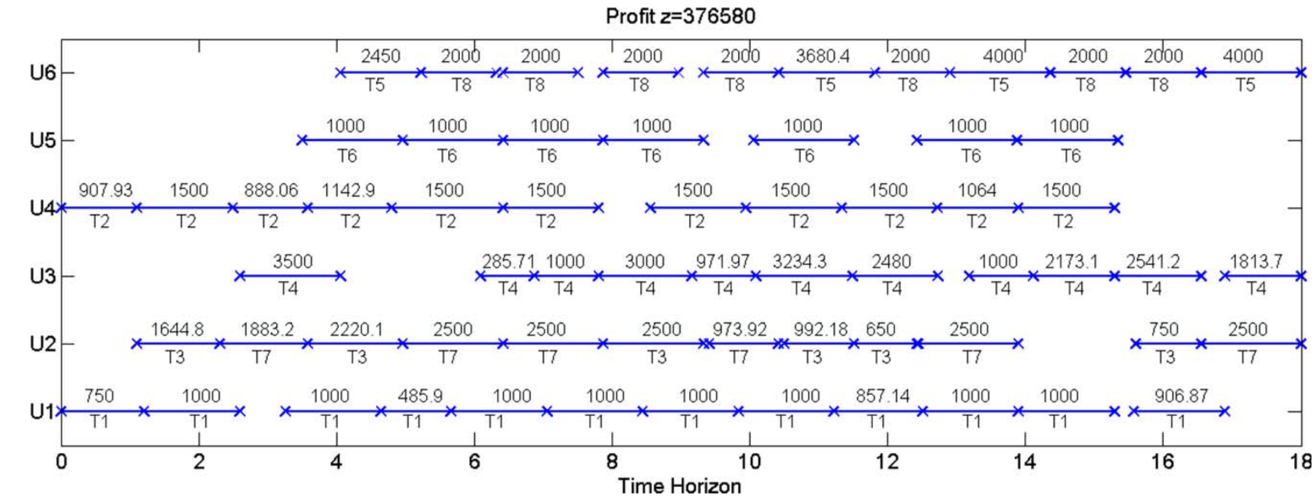

Figure C14. Gantt-chart for Example 5, Case D, with modified two-stage method at $\theta_l^*=0, l=1,2, \theta^* \in CR_{49}$.

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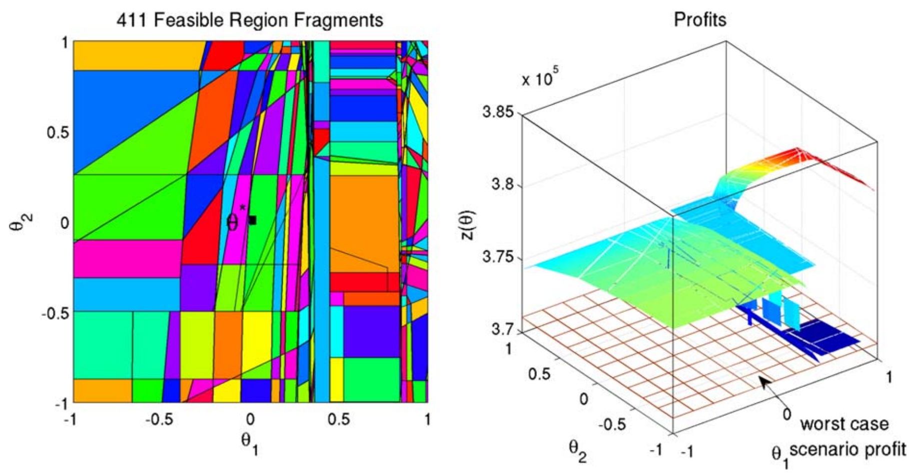

Figure C15. Critical regions and profit with modified two-stage method—Example 5, Case D.

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